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New classes of nonlinear vector coherent states of generalized spin–orbit Hamiltonians

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Abstract

This paper deals with an extension of our previous work (Ben Geloun and Hounkonnou 2007 *J. Phys. A: Math. Theor.* **40** F817) by considering an alternative construction of canonical and deformed vector coherent states (VCSs) of the Gazeau–Klauder type associated with generalized spin–orbit Hamiltonians. We define an annihilation operator which takes into account the finite-dimensional space of states induced by the k-photon transition processes of the two-level atom interacting with the single-mode radiation field. The class of nonlinear VCSs (NVCSs) corresponding to the action of the annihilation operator is deduced and expressed in terms of generalized displacement operators. Various NVCSs including their 'dual' counterparts are also discussed. Also, by using the Hilbert space structure, a new family of NVCSs parametrized by unit vectors of the S^3 sphere has been identified without making use of the annihilation operator.

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1. Introduction

Still attracting the interest of theoreticians, the vector coherent states (VCSs) [1] have been recently studied in statistics [2] and remain relevant in nonlinear quantum optics [3–11]. In particular, their appearance in quantum deformations of physical systems such as generalized spin–orbit Hamiltonians has been proved in [10, 11].

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Let us clarify the precise meaning of 'generalized' spin–orbit interactions according to [11]. In the context of semiconductor physics and spintronics, Rashba [12] and Dresselhaus [13] interactions are typical examples of spin–orbit potentials (for a review of spintronics and spin-Hall effect, see for instance [14]) which can be recast as [15]

$$V_R = \mathrm{i}\mu(a\sigma_- - a^\dagger\sigma_+), \qquad V_D = \lambda(a\sigma_+ + a^\dagger\sigma_-) \tag{1}$$

with coupling constants μ and λ, respectively; a and $a^\dagger$ are the usual Heisenberg generators and, given Pauli-spin matrices σ_1, σ_2 and σ_3, we define $\sigma_\pm := (\sigma_1 \pm \mathrm{i}\sigma_2)/2$. In quantum optics, the Jaynes–Cummings model [16], idealizing the radiation–matter interaction, possesses, in the rotating wave approximation, a spin–orbit interaction which can be written in the same form as V_D [17, 18].

In canonical quantum formulation, a prime notion of nonlinearity for spin–orbit interaction can be probed by considering a k-photon contribution to an intensity-dependent coupling $\lambda(N)$ complex function of the number operator $N = a^\dagger a$. The introduction of such a number-dependent and k-multiphoton coupling becomes significant in the study of the intensity-dependent interaction between a single atom and the radiation field with the atom making k-photon transitions [5, 19–21], as well as in the study of the quantized motion of a single ion in an anharmonic oscillator potential trap [22, 23]. A second stage is reached by adding further nonlinearity by the introduction of f-deformed quantum algebras [24] defined by the modified Heisenberg generators coupled to a free continuous function f of the number operator N such that

$$A^- = af(N), \qquad A^+ = f(N)a^\dagger, \qquad \{N\} = A^+A^- = Nf^2(N), \tag{2}$$

$$[A^-, A^+] = \{N+1\} - \{N\}. \tag{3}$$

Only the limit $f(N) \to 1$ reproduces the ordinary Heisenberg algebra. It was in this framework, introduced by Jannussis *et al* [24], that the earlier notion of 'nonlinear coherent states' (NCSs) was highlighted. NCSs built from a realistic physical model are due to de Matos Filho and Vogel [3] and involve nonclassical properties and quantum interference effects. Afterward, Manko and co-workers [4] interpreted that the f-oscillator action is provided as corresponding to a specific vibration for which the frequency of oscillation becomes energy dependent. Recently, many developments involving the algebra (3) have been made in special function theory, quantum groups and generalized coherent state (CS) quantization (see [25, 26] and references therein).

An f-deformed quantum version of $V_{R,D}$ (1), incorporating the afore mentioned nonlinearities, can be written as

$$\Pi_{k,\varepsilon} = \mathcal{B}^+_{k,\varepsilon}\sigma_+ + \mathcal{B}^-_{k,\varepsilon}\sigma_-, \qquad \mathcal{B}^+_{k,\varepsilon} := A^{-\varepsilon k}\lambda^{\varepsilon}(N), \qquad \mathcal{B}^-_{k,\varepsilon} := \lambda^{-\varepsilon}(N)A^{\varepsilon k}, \tag{4}$$

where the symbol $\varepsilon = \pm$ fixes the notation as $\lambda^+(N) := \lambda(N)$ and $\lambda^-(N) := \overline{\lambda(N)}$, with the bar denoting complex conjugation.

The canonical spin–orbit Hamiltonian includes a dynamical part of a photon field of frequency ω and a Zeeman spin term of atomic frequency ω_0 [15]. It finds henceforth a nonlinear extension regarding deformed spin–orbit Hamiltonians of the reduced (dimensionless) form [11]

$$\mathcal{H}^{\mathrm{red}}_{k,\varepsilon} = \frac{(1+\epsilon)}{2}(\{N+1\} + \{N\}) + \frac{1}{2}(\{N+1\} - \kappa\{N\})\sigma_3 + \mathcal{B}^+_{k,\varepsilon}\sigma_+ + \mathcal{B}^-_{k,\varepsilon}\sigma_-. \tag{5}$$

The ratio $(1+\epsilon) = \omega/\omega_0$ determines the rotating wave approximation if and only if the detuning parameter ϵ satisfies $|\epsilon| \ll 1$ and $|\omega - \omega_0| \ll \omega, \omega_0$. The real parameter κ is introduced in order to recover some known models. A list of significant reduced models connected with (5), its canonical limits $f(N) \to 1$ and $\kappa \to 1$ and applications in quantum optics, in condensed matter physics and in semiconductor physics, in particular, in the so-called domain of spintronics studying some new spin-dependent phenomena in order to build a new generation of electronic devices, are available in [11] and references therein.

We shall refer to the model defined by $\mathcal{H}^{\rm red}_{k,\varepsilon}$ as the $(k, \varepsilon, \kappa, f)$ model, or more simply, as the f-deformed model. This model has an exactly solvable spectrum and its Hilbert space of eigenstates can be decomposed into a finite sequence of k initial states related to k-photon processes and two infinite sequences of states commonly named 'towers'.

In the previous studies [10, 11], classes of VCSs and nonlinear VCSs (NVCSs) have been defined for spin–orbit Hamiltonians associated with a nontrivial action of the annihilation operator only on the two towers. Furthermore, these families of NVCSs meet all requirements of Gazeau–Klauder [27, 28] with the main vector character formulated in terms of the unit sphere S^2 vectors [10] or labeled by two spin states [11]. The case of annihilation operator matrix eigenvalue problem (with diagonal and quaternion matrices) for NVCSs was successfully treated. However, these investigations have ignored the initial states induced by the k-transition processes, with the annihilation operator canceling them by definition. It could be interesting then to ask if NVCSs may include the finite-dimensional space spanned by this limited sequence of states. Moreover, as far as we can establish, deformed displacement operators which could generate NVCSs have not been addressed in prior developments. Recalling that displacement operators for VCSs over matrix domains have been defined by Ali *et al* [1] and also for NCSs ([6, 7, 29] and references therein), one could investigate their form in the case of NVCSs.

In this paper, we give a new construction of NVCSs (including canonical) of the Gazeau–Klauder type associated with generalized spin–orbit Hamiltonians. We define an annihilation operator which takes into account the finite-dimensional space of states of the initial k-photon processes. The class of NVCSs corresponding to the action of the annihilation operator is expressed in terms of generalized displacement operators. Issues concerning 'dual' VCSs and T operators [7] are tackled and exactly solved for certain families of parameters. Besides, a new family of NVCSs parametrized by S^3 unit vectors with an exact resolution of the identity has been identified without making use of the annihilation operator. The latter class involves the definition of CSs identified onto the finite-dimensional Hilbert space [30, 31].

In summary, this paper addresses the following new results:

(i) A generalization of previous constructions of Gazeau–Klauder S^2 and matrix NVCSs for spin–orbit Hamiltonians to Gazeau–Klauder type S^2, normal and quaternionic NVCSs by including the finite sequence of states induced by the k-photon processes in the definition of the generalized deformed Barut–Girardello eigenvalue problem. This allows us to solve properly the issue of discontinuity of NVCSs previously observed at $z = 0$ in [10, 11, 17].

(ii) The explicit identification of novel classes of solvable NVCSs (canonical, generalized (p, q)-deformed in the sense of [25, 26]) with an exact resolution of the identity characterized by a new set of deformation parameters covering all related previous results.

(iii) An extension to the normal matrix domain of the NVCSs with a nontrivial resolution of the identity requiring an integration over a $U(2)$ group. This provides a concrete realization of the formulation by Ali *et al* [1] in nonlinear deformed models.

(iv) The introduction of the concept of deformed displacement operators, deformed dual states (with a peculiar temporal stability) and deformed T operators for NVCSs and matrix NVCSs.

(v) The identification of a large class of NVCSs and their dual counterparts (to be classified) on the basis of the operator ordering occurring in the construction.

(vi) The determination of a new class of S^3 NVCSs still defined in the full Hamiltonian Hilbert space by assigning a new angle to the k-first states.

This paper is organized as follows. Section 2 recalls, in brief, considerations and the Hilbert space structure for classes of spin–orbit models. A quick review of the NVCSs associated with this Hilbert space is also given. Section 3 is devoted to the Hilbert space reorganization and to the definition of ladder operators. Sections 4–6 deal with the construction of a family of Gazeau–Klauder NVCSs defined by the action of the annihilation operator and constituting a generalization of anterior NVCSs of spin–orbit models. Generalized displacement operators, dual states and associated T operators are also treated therein. A different family of S^3 NVCSs is investigated in section 7. A conclusion is given in section 8 and an appendix yields useful relations concerning (p,q)-deformed exponential functions.

2. Nonlinear vector coherent states of spin–orbit models: a quick review

In this section, a rapid overview of previous results on canonical and NVCSs drawn from [10, 11] is given. We aim at generalizing these results in the remaining sections.

As a matter of clarity, let us briefly recall that the bosonic algebra with parameter $\varepsilon = \pm$ is generated by the Heisenberg operators $a^- := a$ and $a^+ := a^\dagger$ so that a^ε is well defined and $a^{-\varepsilon}$ denotes its adjoint. The Heisenberg–Fock algebra then reads off $[a^\varepsilon, a^{-\varepsilon}] = -\varepsilon$. For any $k \in \mathbb{N}$, one defines $a^{\varepsilon k} := (a^\varepsilon)^k$. These operators act on the Fock representation space $F = \{|n\rangle, n \in \mathbb{N}\}$ in the usual manner. We have, by a simple recurrence, for any $k \in \mathbb{N}$,

$$a^{\varepsilon k}|n\rangle = \begin{cases} \left(\dfrac{(n+\varepsilon k)!}{n!}\right)^{\varepsilon/2} |n+\varepsilon k\rangle, & \text{if} \quad \varepsilon = +, \quad n \geqslant 0 \quad \text{or} \quad \varepsilon = -, \quad n \geqslant k \\ 0, & \text{if} \quad \varepsilon = -, \quad n < k. \end{cases} \tag{6}$$

Given a function $g = g(N)$, by action on the representation space, the following ε-commutation rules hold

$$a^\varepsilon g(N) = g(N-\varepsilon)a^\varepsilon, \qquad a^{\varepsilon k} g(N) = g(N-\varepsilon k)a^{\varepsilon k}, \qquad k \in \mathbb{N}. \tag{7}$$

Let $f = f(N)$ be a fixed nonvanishing operator. Then, we define $A^- := a^- f(N)$ and $A^+ := f(N)a^+$. Hence, the notations such that A^ε and, for $k \in \mathbb{N}$, $A^{\varepsilon k} := (A^\varepsilon)^k$ make sense. One expands $A^{\varepsilon k}$ as

$$A^{\varepsilon k} = \left(\frac{f(N)!}{f(N-\varepsilon k)!}\right)^\varepsilon a^{\varepsilon k} = a^{\varepsilon k}\left(\frac{f(N+\varepsilon k)!}{f(N)!}\right)^\varepsilon, \tag{8}$$

where the formal operator $f(N)!$ acts by representation as $f(N)!|n\rangle = f(n)!|n\rangle$ with $f(n)! := f(n)(f(n-1)!)$ and by convention $f(0)! = 1$. Similarly to (7), the identity $A^{\varepsilon k} g(N) = g(N-\varepsilon k)A^{\varepsilon k}$ is true.

The f-deformed oscillator algebra of Jannussis *et al* [24] defined by

$$\begin{aligned} &A^- = af(N), \qquad A^+ = f(N)a^\dagger, \qquad \{N\} := A^+A^- = Nf^2(N), \\ &[A^-, A^+] = (N+1)f^2(N+1) - Nf^2(N) = \{N+1\} - \{N\} \end{aligned} \tag{9}$$

can be represented onto the Fock Hilbert space F as follows:

$$\begin{aligned} &A^-|0\rangle = 0, \qquad A^-|n\rangle = \sqrt{\{n\}}|n-1\rangle, \\ &A^+|n\rangle = \sqrt{\{n+1\}}|n+1\rangle, \qquad \{N\}|n\rangle = \{n\}|n\rangle, \end{aligned} \tag{10}$$

where the deformed number denoted by the symbol $\{n\} := nf^2(n)$ is usually called the f-basic number. One defines $\{0\} := \lim_{n\to 0} nf^2(n)$ and the generalized factorial $\{n\}! := \{n\}(\{n-1\}!)$ with $\{0\}! = 1$ by convention.

Given the Hamiltonian (5), its energy spectrum can be worked out by the usual tangent technique or by the quasideterminant approach [32]. The Hamiltonian Hilbert space $\mathcal{V}$ can be described by a direct sum of a finite set of one-dimensional complex spaces $\mathcal{V}_q$ generated by the orthonormalized states $|E_q^*\rangle$, $q = 0, 1, \ldots, k-1$, $k \geqslant 1$ and infinite-dimensional complex space $\overline{\mathcal{V}}$ spanned by two towers of orthonormalized states $\left|E_n^\pm\right\rangle$, $n \in \mathbb{N}$ such that $n + k\varepsilon \geqslant 0$. Hence $\mathcal{V} = \oplus_{q=0}^{k-1}\mathcal{V}_q \oplus \overline{\mathcal{V}}$ and the reduced Hamiltonian (5) (omitting henceforth the low indices) admits the spectral decomposition

$$\mathcal{H}^{\text{red}} = \sum_{q=0}^{k-1} |E_q^*\rangle E_q^* \langle E_q^*| + \sum_{n=0,\pm}^{\infty} \left|E_{\widetilde{n}}^\pm\right\rangle E_{\widetilde{n}}^\pm \left\langle E_{\widetilde{n}}^\pm\right|, \tag{11}$$

where $\widetilde{n} = n + n_0^\varepsilon$, $n_0^\varepsilon := \max(0, -k\varepsilon)$, and, for $1 \leqslant q \leqslant k-1$,

$$E_q^* = \tfrac{1}{2}[(1+\epsilon-\varepsilon)\{q+1\} + (1+\epsilon+\varepsilon\kappa)\{q\}], \qquad |E_q^*\rangle = |q, -\varepsilon\rangle, \tag{12}$$

while the eigenenergies appearing in the infinite sum are defined as

$$\begin{aligned}
\mathcal{E}(\{n\}) &= \frac{1}{2}\left[\frac{\epsilon}{2}\{n+k\varepsilon+1\} + \frac{1}{2}(1+\epsilon+\kappa)\{n+k\varepsilon\} - \left(1+\frac{\epsilon}{2}\right)\{n+1\} - \frac{1}{2}(1+\epsilon-\kappa)\{n\}\right], \\
Q(\{n\}) &= \left[\mathcal{E}^2(\{n\}) + |\lambda(n+k\varepsilon)|^2\left(\frac{\{n+k\varepsilon\}!}{\{n\}!}\right)^{\varepsilon}\right]^{\frac{1}{2}}, \qquad (13)\\
E_n^\pm &= \frac{1}{2}\left[\frac{\epsilon}{2}\{n+k\varepsilon+1\} + \frac{1}{2}(1+\epsilon+\kappa)\{n+k\varepsilon\} + \left(1+\frac{\epsilon}{2}\right)\{n+1\}\right.\\
&\quad \left.+ \frac{1}{2}(1+\epsilon-\kappa)\{n\}\right] \pm Q(\{n\}). \qquad (14)
\end{aligned}$$

Given

$$\sin\vartheta(\{n\}) = \mathrm{e}^{\mathrm{i}\varphi_\lambda(n)}\left[\frac{Q(\{n\}) - \mathcal{E}(\{n\})}{2Q(\{n\})}\right]^{\frac{1}{2}}, \qquad \cos\vartheta(\{n\}) = \left[\frac{Q(\{n\}) + \mathcal{E}(\{n\})}{2Q(\{n\})}\right]^{\frac{1}{2}}, \tag{15}$$

the eigenstates of the two towers can be expressed by

$$\left|E_n^+\right\rangle = \sin\vartheta(\{n\})|n, +\rangle + \cos\vartheta(\{n\})|n+k\varepsilon, -\rangle, \tag{16}$$

$$\left|E_n^-\right\rangle = \cos\vartheta(\{n\})|n, +\rangle - \overline{\sin\vartheta(\{n\})}|n+k\varepsilon, -\rangle, \tag{17}$$

with $n \geqslant k$ for $\varepsilon = -$. The notation $\overline{X}$ stands for the complex conjugate of the quantity X. Note that $\Delta E_n = E_n^+ - E_n^- = 2Q(\{n\})$ assigns the Zeeman spin splitting to the Rabi frequency up to a constant.

Allowing the passage from one basis of $\mathcal{V}$ to another, the operators

$$\mathcal{U} = \sum_{n=0,\pm}^{\infty} \left|E_{\widetilde{n}}^\pm\right\rangle\langle n, \pm|, \qquad \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |n, \pm\rangle\left\langle E_{\widetilde{n}}^\pm\right| \tag{18}$$

are mutually adjoint on $\overline{\mathcal{V}}$ but nonunitary on $\mathcal{V}$. We have the identities $\mathcal{U}^\dagger\mathcal{U} = \mathbb{I}_{\mathcal{V}}, \mathcal{U}\mathcal{U}^\dagger = \mathbb{I}_{\overline{\mathcal{V}}}, \mathcal{U}\mathcal{V} = \overline{\mathcal{V}}, \mathcal{U}^\dagger\mathcal{V} = \mathcal{V}$ and $\mathcal{U}^\dagger\overline{\mathcal{V}} = \mathcal{V}$. Therefore, the reduced Hamiltonian (11) can be diagonalized in terms of

$$\mathbb{H}^{\text{red}} = \mathcal{U}^\dagger\mathcal{H}^{\text{red}}\mathcal{U} = \sum_{n=0,\pm}^{\infty} |n, \pm\rangle E_{\widetilde{n}}^\pm \langle n, \pm|, \tag{19}$$

$$\mathcal{U}\mathbb{H}^{\mathrm{red}}\mathcal{U}^{\dagger} = \sum_{n=0,\pm}^{\infty} \big|E^{\pm}_{\widetilde{n}}\big\rangle E^{\pm}_{\widetilde{n}}\big\langle E^{\pm}_{\widetilde{n}}\big| = \mathcal{H}^{\mathrm{red}} - \sum_{q=0}^{k-1} |E^{*}_{q}\rangle E^{*}_{q}\langle E^{*}_{q}|. \tag{20}$$

The following operators are adjoint of one another on $\mathcal{V}$:

$$\mathbb{M}^{-} = \sum_{n=0,\pm}^{\infty} |n-1,\pm\rangle K_{\pm}(\{n\})\langle n,\pm|, \qquad \mathbb{M}^{+} = \sum_{n=0,\pm}^{\infty} |n+1,\pm\rangle \overline{K_{\pm}(\{n+1\})}\langle n,\pm|, \tag{21}$$

with $K_{\pm}(\{n\})$ being the arbitrary complex functions of $\{n\}$ such that $K_{\pm}(\{0\}) = 0$. Reciprocally, the operators $\mathcal{M}^{-} = \mathcal{U}\mathbb{M}^{-}\mathcal{U}^{\dagger}$ and $\mathcal{M}^{+} = \mathcal{U}\mathbb{M}^{+}\mathcal{U}^{\dagger}$ are mutually adjoint on the subspace $\overline{\mathcal{V}}$ and have lowering and raising actions within each of the towers, namely,

$$\mathcal{M}^{-}|E^{*}_{q}\rangle = 0, \qquad \mathcal{M}^{+}|E^{*}_{q}\rangle = 0, \qquad q = 0,1,\ldots,k-1, \tag{22}$$

$$\mathcal{M}^{-}\big|E^{\pm}_{\widetilde{n}}\big\rangle = K_{\pm}(\{n\})\big|E^{\pm}_{\widetilde{n-1}}\big\rangle, \qquad \mathcal{M}^{+}\big|E^{\pm}_{\widetilde{n}}\big\rangle = \overline{K_{\pm}(\{n+1\})}\big|E^{\pm}_{\widetilde{n+1}}\big\rangle. \tag{23}$$

The generalized annihilation operator eigenvalue problem

$$\mathcal{M}^{-}|z;\tau_{\pm};\theta,\phi\rangle = z\widetilde{\mathbb{Q}}_{\mathcal{V}}|z;\tau_{\pm};\theta,\phi\rangle, \tag{24}$$

$$\widetilde{\mathbb{Q}}_{\mathcal{V}} := \sum_{q=0}^{k-1} |E^{*}_{q}\rangle\langle E^{*}_{q}| + \sum_{n=0,\pm}^{\infty} \big|E^{\pm}_{\widetilde{n}}\big\rangle h^{\pm}_{f}(n)\big\langle E^{\pm}_{\widetilde{n}}\big|, \tag{25}$$

where the quantities $h^{\pm}_{f}(n) \neq 0$ are such that $h^{\pm}_{f}(n) \to 1$ as $f(N) \to 1$, should be solved in order to define the (k,ε,κ,f)-VCS denoted by $|z;\tau_{\pm};\theta,\phi\rangle$. The terminology *generalized* is justified by the fact that the eigenvalue problem is stated here through generalized deformed arbitrary structure functions $K_{\pm}$ and $h^{\pm}_{f}$, incorporated in $\mathcal{M}^{-}$ and $\widetilde{\mathbb{Q}}_{\mathcal{V}}$, respectively, and encompasses known related eigenvalue problems [17] (the matrix eigenvalue problem developed in the following includes the models in [1] (see also references therein)). As the most simple instance, at the canonical limit $\lim_{f(N)\to 1}\widetilde{\mathbb{Q}}_{\mathcal{V}} \equiv \mathbb{I}_{\mathcal{V}}$, the problem (24) exactly reduces to a Barut–Girardello eigenvalue problem for VCSs. In addition, the parameters $\tau_{\pm}$ are introduced for the Gazeau–Klauder axiom of temporal stability, (θ,ϕ) parametrize the unit vectors of the sphere S^2 and also determine the vector feature of NVCSs $|z;\tau_{\pm};\theta,\phi\rangle$.

The general form of the (k,ε,κ,f)-VCSs fulfilling all axioms of Gazeau–Klauder is given by [11]

$$\begin{aligned}|z;\tau_{\pm};\theta,\phi\rangle = {}& \mathcal{N}^{+}(|z|)\cos\theta \sum_{n=0}^{\infty} \frac{z^n}{K^{0}_{+}(\{n\})!}\big(h^{+}_{f}(n-1)!\big)h^{+}_{f}(0)\,\mathrm{e}^{-\mathrm{i}\omega_0\tau_{+}E^{+}_{\widetilde{n}}}\big|E^{+}_{\widetilde{n}}\big\rangle \\ & + \mathcal{N}^{-}(|z|)\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta \sum_{n=0}^{\infty} \frac{z^n}{K^{0}_{-}(\{n\})!}\big(h^{-}_{f}(n-1)!\big)h^{-}_{f}(0)\,\mathrm{e}^{-\mathrm{i}\omega_0\tau_{-}E^{-}_{\widetilde{n}}}\big|E^{-}_{\widetilde{n}}\big\rangle,\end{aligned} \tag{26}$$

$$\mathcal{N}^{\pm}(|z|) = \left[\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\big(K^{0}_{\pm}(\{n\})!\big)^2}\Big(\big(h^{\pm}_{f}(n-1)!\big)h^{\pm}_{f}(0)\Big)^2\right]^{-1/2}, \tag{27}$$

where $\mathcal{N}^{\pm}(|z|)$ are normalization factors, the real positive functions $K^{0}_{\pm}(\{n\}) := |K_{\pm}(\{n\})|$ and $h^{\pm}_{f}(n)$ are yet to be specified; $K^{0}_{\pm}(\{n\})! := \prod_{k=1}^{n} K^{0}_{\pm}(\{k\})$, $h^{\pm}_{f}(n-1)! := \prod_{k=1}^{n-1} h^{\pm}_{f}(k)$, with, by convention, $K^{0}_{\pm}(\{0\})! = 1$, $h^{\pm}_{f}(0)! = 1$ and $h^{\pm}_{f}(-1)! = \big(h^{\pm}_{f}(0)\big)^{-1}$. The convergence radii of $\mathcal{N}^{\pm}(|z|)$ are $R_{\pm} = \lim_{n\to\infty}\big(K^{0}_{\pm}(\{n\})/h^{\pm}_{f}(n-1)\big)$ and depend on the choice of

$K^0_\pm(\{n\})$ as well as on $h^\pm_f(n)$. Consequently, the NVCSs of this form live in the disc $|z| \leqslant R$, $R = \min(R_+, R_-)$.

The heretofore developments also meet a matrix formulation. Given the 2×2 diagonal matrix $K(\{n\}) := \mathrm{diag}(K_+(\{n\}), K_-(\{n\}))$ and $\overline{K(\{n\})}$ its adjoint, the diagonal matrix annihilation operator connected with (21) is

$$\mathbb{M}^- = \sum_{n=0,\pm}^{\infty} K(\{n\})|n-1, \pm\rangle\langle n, \pm|, \tag{28}$$

and again $\mathcal{M}^- = \mathcal{U}\mathbb{M}^-\mathcal{U}^\dagger$. We translate the eigenvalue problem (24) equally as

$$\mathcal{M}^-|(z, w); \tau_\pm; \pm\rangle = \widetilde{\mathfrak{Z}}(z, w)\widetilde{\mathbb{Q}}_\mathcal{V}|(z, w); \tau_\pm; \pm\rangle, \tag{29}$$

where $\widetilde{\mathfrak{Z}}(z, w)$ is a 2×2 matrix operator of the two complex variables (z, w), and $\pm$ is the spin-vector dependence now replacing the S^2 unit sphere vectors. Assuming that $\mathfrak{Z} = \mathcal{U}^\dagger\widetilde{\mathfrak{Z}}\mathcal{U}$ is a complex constant matrix and defining $\mathbb{Q}_\mathcal{V} := \mathcal{U}^\dagger\widetilde{\mathbb{Q}}_\mathcal{V}\mathcal{U}$, we rephrase (29) into the diagonal basis as

$$\mathbb{M}^-|\mathfrak{Z}; \tau_\pm; \pm\rangle = \mathfrak{Z}\mathbb{Q}_\mathcal{V}|\mathfrak{Z}; \tau_\pm; \pm\rangle, \qquad |\mathfrak{Z}; \tau_\pm; \pm\rangle = \mathcal{U}^\dagger|(z, w); \tau_\pm; \pm\rangle, \tag{30}$$

$$\mathbb{Q}_\mathcal{V} = \sum_{n=0,\pm}^{\infty} h_f(n)|n, \pm\rangle\langle n, \pm|, \qquad h_f(n) = \mathrm{diag}\big(h^+_f(n), h^-_f(n)\big).$$

Problem (30) is a matrix eigenvalue problem. A canonical class of these problems can be exactly carried out if $\mathfrak{Z}$ is in a normal or in a quaternionic matrix domain [1]. In the particular instance of a nonlinear model, assuming also that $\mathfrak{Z} = \mathrm{diag}(z, w)$, the general Gazeau–Klauder NVCS solution of the eigenvalue problem (30) is given by

$$|\mathfrak{Z}; \tau_\pm; \pm\rangle = N(\mathfrak{Z}) \sum_{n=0}^{\infty} R_0(n) \exp[-\mathrm{i}\omega_0\tau E_{\tilde{n}}]\mathfrak{Z}^n|n, \pm\rangle, \tag{31}$$

$$N(\mathfrak{Z})^{-2} = \sum_{n=0}^{\infty} \left(|z|^{2n}\big(R^0_+(n)\big)^2 + |w|^{2n}\big(R^0_-(n)\big)^2\right), \tag{32}$$

$$R^0(n) = \mathrm{diag}\big(R^0_+(n), R^0_-(n)\big) = (K^0(\{n\})!)^{-1}(h_f(n-1)!h_f(0)), \tag{33}$$

where $N(\mathfrak{Z})$ is the normalization factor, $\tau = \mathrm{diag}(\tau_+, \tau_-)$ and $E_{\tilde{n}} = \mathrm{diag}\big(E^+_{\tilde{n}}, E^-_{\tilde{n}}\big)$. Note that the convergence radii of series (32) are such that $|z| \leqslant L_+, |w| \leqslant L_-$ and $L_\pm = \lim_{n\to\infty} K^0_\pm(\{n\})\big/h^\pm_f(n-1)$.

Given in the form (26) or (31), the NVCSs still contain some undetermined quantities, hence the name of general form of NVCSs. They have been built with respect to three over four Gazeau–Klauder axioms: the continuity in labeling, the temporal stability, the normalizability. It then remains the resolution of the identity. It turns out that explicit examples of deformation of NVCSs in both forms have exact solutions to their Stieljes moment problem by further constraining the operator algebra as $[\mathcal{M}^-, \mathcal{M}^+] = \mathbb{I}_{\overline{\mathcal{V}}}$, or by the action identity constraint [28] in the case of the canonical limit only [10, 11]. Hence, generalized solvable NVCSs meeting all requirements of Gazeau–Klauder have been successfully built from a deformed physical model.

3. Hamiltonian Hilbert space organization and ladder operators

From this section, we start our main new results dealing with an extension of the previous analysis of $(k, \varepsilon, \kappa, f)$-VCSs by a prolongation of the annihilation operator $\mathcal{M}^-$ onto the finite-dimensional space generated by $\oplus_{q=0}^{k-1}\mathcal{V}_q$. A number of deformed displacements for S^2, matrix NVCSs and issues about deformed T operators will be addressed in the following sections. The results of the remaining sections are totally dependent on the Hilbert space organization that we set in this section. We mention that, albeit quantities and operators may differ, same notations as in section 2 will be used hereafter.

Let us first redefine the eigenstates and eigenenergies as

$$\left|e_n^-\right\rangle = |E_n^*\rangle, \qquad e_n^- = E_n^*, \qquad n = 0, 1, \ldots, k-1, \tag{34}$$

$$\left|e_n^-\right\rangle = \left|E^-_{n-n_0^{-\varepsilon}}\right\rangle, \qquad e_n^- = E^-_{n-n_0^{-\varepsilon}}, \qquad n = k, k+1, \ldots, \tag{35}$$

$$\left|e_n^+\right\rangle = |E^+_{n+n_0^{\varepsilon}}\rangle, \qquad e_n^+ = E^+_{n+n_0^{\varepsilon}}, \qquad n = 0, 1, \ldots, \tag{36}$$

so that we obtain two new towers of states $\left|e_n^\pm\right\rangle$, $n = 0, 1, \ldots.$ Note that the finite sequence $|E_n^*\rangle$, $n \leqslant k-1$, could also be added to the tower $\left|e_n^+\right\rangle$ without loss of generality.

The spectral decomposition of the Hamiltonian (11) appears in the simpler form

$$\mathcal{H}^{\mathrm{red}} = \sum_{n=0,\pm}^{\infty} \left|e_n^\pm\right\rangle e_n^\pm \left\langle e_n^\pm\right|. \tag{37}$$

Next the passage operators $\mathcal{U}$ and $\mathcal{U}^\dagger$, such that $\mathcal{U}|n,\pm\rangle = \left|e_n^\pm\right\rangle$ and $\mathcal{U}^\dagger\left|e_n^\pm\right\rangle = |n,\pm\rangle$, encounter the expansion

$$\mathcal{U} = \sum_{n=0,\pm}^{\infty} \left|e_n^\pm\right\rangle\langle n,\pm|, \qquad \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |n,\pm\rangle\left\langle e_n^\pm\right|. \tag{38}$$

They come now as unitary operators on $\mathcal{V}$ since they satisfy

$$\mathcal{U}\mathcal{V} = \mathcal{V}, \qquad \mathcal{U}^\dagger\mathcal{V} = \mathcal{V}, \qquad \mathcal{U}^\dagger\mathcal{U} = \mathbb{I}_{\mathcal{V}} = \mathcal{U}\mathcal{U}^\dagger. \tag{39}$$

Therefore, the Hamiltonian $\mathcal{H}^{\mathrm{red}}$ can be written in a diagonal form

$$\mathbb{H}^{\mathrm{red}} = \mathcal{U}^\dagger\mathcal{H}^{\mathrm{red}}\mathcal{U} = \sum_{n=0,\pm}^{\infty} |n,\pm\rangle e_n^\pm \langle n,\pm|. \tag{40}$$

Conversely, in contrast to (20), we have $\mathcal{U}\mathbb{H}^{\mathrm{red}}\mathcal{U}^\dagger = \mathcal{H}^{\mathrm{red}}$. Thus, this notation improves the understanding of the Hamiltonian Hilbert space with a basis $\left|e_n^\pm\right\rangle$ mapped onto the diagonal basis $|n,\pm\rangle$ via a unitary operator $\mathcal{U}^\dagger$.

Now, let us focus on the annihilation and creation operators. To begin with, consider the diagonal and mutually adjoint operators

$$\mathbb{M}^- = \sum_{n=0,\pm}^{\infty} |n-1,\pm\rangle K_\pm(\{n\})\langle n,\pm|, \qquad \mathbb{M}^+ = \sum_{n=0,\pm}^{\infty} |n+1,\pm\rangle \overline{K_\pm(\{n+1\})}\langle n,\pm|, \tag{41}$$

where again $K_\pm(\{n\})$ are some functions of the f-basic number $\{n\}$ with initial value $K_\pm(\{0\}) = 0$. Changing the basis, the corresponding operators

$$\mathcal{U}\mathbb{M}^-\mathcal{U}^\dagger = \mathcal{M}^- = \sum_{n=0,\pm}^{\infty} \left|e_{n-1}^\pm\right\rangle K_\pm(\{n\})\left\langle e_n^\pm\right|, \tag{42}$$

$$\mathcal{U}\mathbb{M}^{+}\mathcal{U}^{\dagger} = \mathcal{M}^{+} = \sum_{n=0,\pm}^{\infty} \left|e_{n+1}^{\pm}\right\rangle\overline{K_{\pm}(\{n+1\})}\left\langle e_{n}^{\pm}\right|, \tag{43}$$

with lowering and raising actions as, $\forall\, n \in \mathbb{N}$,

$$\mathcal{M}^{-}\left|e_{n}^{\pm}\right\rangle = K_{\pm}(\{n\})\left|e_{n-1}^{\pm}\right\rangle, \qquad \mathcal{M}^{+}\left|e_{n}^{\pm}\right\rangle = \overline{K_{\pm}(\{n+1\})}\left|e_{n+1}^{\pm}\right\rangle, \tag{44}$$

define mutually adjoint ladder operators on $\mathcal{V}$.

In terms of the 'old' basis $\{|E_q^*\rangle, q = 0, 1, \ldots, k-1\} \cup \left\{\left|E_{\tilde{n}}^{\pm}\right\rangle, n = 0, 1, \ldots\right\}$, we have

$$\mathcal{M}^{-}|E_0^*\rangle = 0, \qquad \mathcal{M}^{+}|E_0^*\rangle = K_{-}(\{1\})|E_1^*\rangle, \tag{45}$$

$$\mathcal{M}^{-}|E_q^*\rangle = K_{-}(\{q\})|E_{q-1}^*\rangle, \qquad \mathcal{M}^{+}|E_q^*\rangle = K_{-}(\{q+1\})|E_{q+1}^*\rangle, \qquad q = 1, \ldots, k-2, \tag{46}$$

$$\mathcal{M}^{-}|E_{k-1}^*\rangle = K_{-}(\{k-1\})|E_{k-2}^*\rangle, \qquad \mathcal{M}^{+}|E_{k-1}^*\rangle = K_{-}(\{k\})\left|E_{\tilde{0}}^{-}\right\rangle, \tag{47}$$

$$\mathcal{M}^{-}\left|E_{\tilde{0}}^{-}\right\rangle = K_{-}(\{k\})|E_{k-1}^*\rangle, \qquad \mathcal{M}^{+}\left|E_{\tilde{0}}^{-}\right\rangle = K_{-}(\{1\})\left|E_{\tilde{1}}^{-}\right\rangle \tag{48}$$

$$\mathcal{M}^{-}\left|E_{\tilde{n}}^{\pm}\right\rangle = K_{\pm}(\{n\})\left|E_{\tilde{n}-1}^{\pm}\right\rangle, \qquad \mathcal{M}^{+}\left|E_{\tilde{n}}^{\pm}\right\rangle = \overline{K_{\pm}(\{n+1\})}\left|E_{\tilde{n}+1}^{\pm}\right\rangle, n \geqslant 1, \tag{49}$$

which are to be compared with (22) and (23). The action of the annihilation operator is therefore ensured between the two parts of the Hilbert space $\oplus_{q=0}^{k-1}\mathcal{V}_q$ and $\overline{\mathcal{V}}$ by the operator $|E_{k-1}^*\rangle K_{-}(\{k\})\left\langle E_{\tilde{0}}^{-}\right|$ (and its adjoint in the case of raising action). It could also be defined over the type of prolongation of the annihilation operator using the mapping $|E_0^*\rangle\langle E_{k-1}^*|$ and entailing a kind of 'cyclic' annihilation operator onto $\oplus_{q=0}^{k-1}\mathcal{V}_q$. This type of annihilation operator proves to be well defined for finite-dimensional Hilbert spaces [30, 31]. However, restricted to that latter situation, some difficulties emerge in the construction of CSs as we shall define it later.

Finally, we give the matrix entries of the annihilation operator $\mathcal{M}^{-}$ (42) in the diagonal basis $\{|n, \pm\rangle, n \in \mathbb{N}\}$

$$\begin{aligned}\mathcal{M}^{-} = \sum_{q=0}^{k-1} |q-1, -\varepsilon\rangle\mathcal{M}^{-}_{-\varepsilon-\varepsilon}(\{q\})\langle q, -\varepsilon| + |k-1, -\varepsilon\rangle\mathcal{M}^{-}_{-\varepsilon+}(\{k\})\langle\widetilde{0}, +| \\ + |k-1, -\varepsilon\rangle\mathcal{M}^{-}_{-\varepsilon-}(\{k\})\langle\widetilde{0} + k\varepsilon, -| + \sum_{n=0}^{\infty} |\widetilde{n}, +\rangle\mathcal{M}^{-}_{++}(\{n\})\langle\widetilde{n}+1, +| \\ + \sum_{n=0}^{\infty} |\widetilde{n}, +\rangle\mathcal{M}^{-}_{+-}(\{n\})\langle\widetilde{n} + k\varepsilon + 1, -| + \sum_{n=0}^{\infty} |\widetilde{n} + k\varepsilon - 1, -\rangle\mathcal{M}^{-}_{-+}(\{n\})\langle\widetilde{n}, +| \\ + \sum_{n=0}^{\infty} |\widetilde{n} + k\varepsilon - 1, -\rangle\mathcal{M}^{-}_{--}(\{n\})\langle\widetilde{n} + k\varepsilon, -|\end{aligned} \tag{50}$$

where

$$\begin{aligned}&\mathcal{M}^{-}_{-\varepsilon-\varepsilon}(\{0\}) = 0, \qquad \mathcal{M}^{-}_{-\varepsilon-\varepsilon}(\{q\}) = K_{-}(\{q\}), \qquad q = 1, \ldots, k-1,\\ &\mathcal{M}^{-}_{-\varepsilon+}(\{k\}) = K_{-}(\{k\})\cos\vartheta(\{\widetilde{0}\}), \qquad \mathcal{M}^{-}_{-\varepsilon-}(\{k\}) = -K_{-}(\{k\})\sin\vartheta(\{\widetilde{0}\}),\\ &\mathcal{M}^{-}_{++}(\{n\}) = \sin\vartheta(\{\widetilde{n}\})\overline{\sin\vartheta(\{\widetilde{n}+1\})}K_{+}(\{n+1\}) + \cos\vartheta(\{\widetilde{n}\})\cos\vartheta(\{\widetilde{n}+1\})K_{-}(\{n+1\}),\\ &\mathcal{M}^{-}_{+-}(\{n\}) = \sin\vartheta(\{\widetilde{n}\})\cos\vartheta(\{\widetilde{n}+1\})K_{+}(\{n+1\}) - \cos\vartheta(\{\widetilde{n}\})\sin\vartheta(\{\widetilde{n}+1\})K_{-}(\{n+1\}),\\ &\mathcal{M}^{-}_{-+}(\{n\}) = \cos\vartheta(\{\widetilde{n}-1\})\overline{\sin\vartheta(\{\widetilde{n}\})}K_{+}(\{n\}) - \overline{\sin\vartheta(\{\widetilde{n}-1\})}\cos\vartheta(\{\widetilde{n}\})K_{-}(\{n\}),\\ &\mathcal{M}^{-}_{--}(\{n\}) = \cos\vartheta(\{\widetilde{n}-1\})\cos\vartheta(\{\widetilde{n}\})K_{+}(\{n\}) + \overline{\sin\vartheta(\{\widetilde{n}-1\})}\sin\vartheta(\{\widetilde{n}\})K_{-}(\{n\}),\end{aligned} \tag{51}$$

with $\mathcal{M}^{-}_{-+}(\{0\}) = 0 = \mathcal{M}^{-}_{--}(\{0\})$, for $K_{\pm}(0) = 0$. Taking the complex conjugate of these expressions leads to the analog of these quantities connected with the raising operator. The parameter functions $K_{\pm}(\{n\})$ again indicate the freedom in the choice of creation and annihilation operators that we will restrict in order to allow the existence of NVCSs according to a series of general axioms [27]. As was observed in [11], the current Hilbert space structure and its organization do not depend on the quantization scheme. Thus, the above results remain valid for the undeformed canonical situation recovered by setting $\{n\} \to n$ in all expressions.

4. Nonlinear vector coherent states

In this section, we construct classes of Gazeau–Klauder NVCSs using the annihilation operator eigenvalue problem of forms (24) and (29) resolved now within the new Hilbert space structure. The Hamiltonian expectation value and the Rabi oscillations for these states are computed and exact solutions to their overcompleteness problem are given. The corresponding matrix formulation, deformed displacement operators, dual NVCSs and T operators will be consistently defined in the following sections.

4.1. Identifying NVCSs

We proceed to the construction of NVCSs with respect to the set of Gazeau–Klauder axioms by solving the generalized eigenvalue problem

$$\mathcal{M}^{-}|z;\tau_{\pm};\theta,\phi\rangle = z\widetilde{\mathbb{Q}}_{\mathcal{V}}|z;\tau_{\pm};\theta,\phi\rangle, \tag{52}$$

$$\widetilde{\mathbb{Q}}_{\mathcal{V}} := \sum_{n=0,\pm}^{\infty} |e_n^{\pm}\rangle h_f^{\pm}(n)\langle e_n^{\pm}|, \tag{53}$$

where the parameters $\tau_{\pm}$, θ and ϕ are introduced below and the functions $h_f^{\pm}(n)$ have the same properties as stated before. Assuming that the solution of the problem (52) is of the form

$$|z;\tau_{\pm};\theta,\phi\rangle = \sum_{n=0,\pm}^{\infty} C_n^{\pm}(z)\big|e_n^{\pm}\big\rangle, \tag{54}$$

where $C_n^{\pm}(z)$ are complex continuous functions of the complex variable z, and then substituting (54) into (52), leads to the recurrence relation

$$\forall\, n \in \mathbb{N}, \qquad C_{n+1}^{\pm}(z)K_{\pm}(\{n+1\}) = zh_f^{\pm}(n)C_n^{\pm}(z). \tag{55}$$

A simple solution of this recurrence is

$$C_n^{\pm}(z) = \frac{z^n}{K_{\pm}(\{n\})!}\big(h_f^{\pm}(n-1)!\big)h_f^{\pm}(0)C_0^{\pm}(z), \qquad n \geqslant 1, \tag{56}$$

with $C_0^{\pm}(z)$ being the arbitrary continuous complex functions of z. The sense of generalized factorials remains as $K_{\pm}(\{n\})! := \prod_{p=1}^{n} K_{\pm}(\{p\}), h_f^{\pm}(n-1)! = \prod_{p=1}^{n-1} h_f^{\pm}(p)$ with, by convention, $K_{\pm}(\{0\})! = 1, h_f^{\pm}(0)! = 1$ and $h_f^{\pm}(-1)! = (h_f^{\pm}(0))^{-1}$. Therefore, the expression of $C_n^{\pm}(z)$ (56) is still correct for $n \geqslant 0$.

Temporal stability condition refers to the CS stability under time evolution operator $U(t) = \exp(-\mathrm{i}\omega_0 t\mathcal{H}^{\mathrm{red}})$. In other words, the CSs should transform into one another under time translations. More subtle considerations about this notion can be found in [8]. The latter axiom can be reached by introducing a phase $\varphi_{\pm}(\{n\})$ such as $K_{\pm}(\{n\}) =$

$\exp[\mathrm{i}\varphi_{\pm}(\{n\})]K^0_{\pm}(\{n\})$, $K^0_{\pm}(\{n\})$ being real positive quantities, and after imposing the relations, for all $n = 1, 2, \ldots$,

$$\varphi_{\pm}(\{n\}) = \omega_0\tau_{\pm}\left[e^{\pm}_n - e^{\pm}_{n-1}\right], \tag{57}$$

where $\tau_{\pm}$ is a new parameter. One has to define $C^{\pm}_0(z) = \mathcal{C}^{\pm}_0(z)\exp\left[-\mathrm{i}\omega_0\tau_{\pm}e^{\pm}_0\right]$ so that the problem (52) gives the solution

$$|z;\tau_{\pm};\theta,\phi\rangle = \sum_{n=0,\pm}^{\infty}\frac{z^n}{K^0_{\pm}(\{n\})!}\left(h^{\pm}_f(n-1)!\right)h^{\pm}_f(0)\mathcal{C}^{\pm}_0(z)\,\mathrm{e}^{-\mathrm{i}\omega_0\tau_{\pm}e^{\pm}_n}\left|e^{\pm}_n\right\rangle \tag{58}$$

with the property

$$U(t)|z;\tau_{\pm};\theta,\phi\rangle = |z;\tau_{\pm}+t;\theta,\phi\rangle. \tag{59}$$

At this stage, we mention that the construction adopted here to ensure the temporal stability requirement is equivalent to the procedure given by Roknizadeh and Tavassoly [8] where an evolution operator maps any generalized CS to a temporal stable one. We also point out the fact that the anterior problems of a singular state associated with the eigenvalue $z = 0$ [10, 11, 17] have been totally removed. For instance, in the case $k = 1$, i.e. the so-called Jaynes–Cummings model, the states $|E^*\rangle$ and $\left|E^{\pm}_0\right\rangle$ share the same eigenvalue $z = 0$. However, the state $|E^*\rangle$ is not included in the definition of the CSs; then a singularity breaks the Gazeau–Klauder axiom of *continuity of labeling* in z for the eigenvalue problem (24). In the case of the k-multiphoton model, k states violate the latter axiom. This difficulty is generally circumvented by the simple claim that the vectors with eigenvalue $z = 0$ may be expressed as a combination of the eigenstates $\left|E^{+}_{n^{\varepsilon}_0}\right\rangle$ and $\left|E^{-}_{n^{\varepsilon}_0}\right\rangle$. Here such an issue is totally avoided due to the 'continuous' action of the annihilation operator, passing from the infinite towers to the finite-dimensional part of the Hamiltonian Hilbert space.

Defining

$$\mathcal{C}^{+}_0(z) = \mathcal{N}^{+}(|z|)\cos\theta, \qquad \mathcal{C}^{-}_0(z) = \mathcal{N}^{-}(|z|)\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta, \tag{60}$$

introducing the S^2 unit vector coordinates (θ,ϕ), and the functions

$$\mathcal{N}^{\pm}(|z|) = \left[\sum_{n=0}^{\infty}\frac{|z|^{2n}}{\left(K^0_{\pm}(\{n\})!\right)^2}\left(\left(h^{\pm}_f(n-1)!\right)h^{\pm}_f(0)\right)^2\right]^{-1/2} \tag{61}$$

of convergence radii $R_{\pm}$

$$R_{\pm} = \lim_{n\to\infty}\left[\frac{K^0_{\pm}(\{n\})}{h^{\pm}_f(n-1)}\right] \tag{62}$$

ensures the *normalization axiom*. Finally, the general (k,ε,κ,f)-VCS associated with the generalized spin–orbit model (37) has the form

$$\begin{aligned}|z;\tau_{\pm};\theta,\phi\rangle &= \mathcal{N}^{+}(|z|)\cos\theta\sum_{n=0}^{\infty}\frac{z^n}{K^0_{+}(\{n\})!}\left(h^{+}_f(n-1)!\right)h^{+}_f(0)\,\mathrm{e}^{-\mathrm{i}\omega_0\tau_{+}e^{+}_n}\left|e^{+}_n\right\rangle\\ &\quad+\mathcal{N}^{-}(|z|)\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta\sum_{n=0}^{\infty}\frac{z^n}{K^0_{-}(\{n\})!}\left(h^{-}_f(n-1)!\right)h^{-}_f(0)\,\mathrm{e}^{-\mathrm{i}\omega_0\tau_{-}e^{-}_n}\left|e^{-}_n\right\rangle,\end{aligned} \tag{63}$$

with $|z| \leqslant R$, $R = \min(R_+, R_-)$; the positive functions $K^0_{\pm}(\{n\})$ and $h^{\pm}_f(n)$ are to be specified. One notices the similar structure of NVCSs (63) and (26). However, these two classes of CSs radically differ since they are not built onto the same Hilbert space.

The *overcompleteness condition* is a necessary axiom that any family of CSs ought to satisfy [1, 27, 28]. For the NVCSs (63), this condition can be formulated as

$$\mathbb{I}_{\mathcal{V}} = \sum_{n=0,\pm}^{\infty} |e_n^{\pm}\rangle\langle e_n^{\pm}| = \int_{D_R\times S^2} \mathrm{d}\mu(z;\theta,\phi)|z;\tau_{\pm};\theta,\phi\rangle\langle z;\tau_{\pm};\theta,\phi|, \tag{64}$$

with the $SU(2)$ matrix-valued integration measure over $D_R \times S^2$, $\mathrm{d}\mu(z;\theta,\phi)$ admitting the parametrization as

$$\mathrm{d}\mu(z;\theta,\phi) = \mathrm{d}^2z\,\mathrm{d}\theta\sin\theta\,\mathrm{d}\phi\left\{\mathcal{W}^{+}(|z|)\sum_{n=0}^{\infty}|e_n^{+}\rangle\langle e_n^{+}| + \mathcal{W}^{-}(|z|)\sum_{n=0}^{\infty}|e_n^{-}\rangle\langle e_n^{-}|\right\}, \tag{65}$$

where $\mathcal{W}^{\pm}(|z|)$ are yet unknown real weight functions. A direct substitution in (64), using the radial parametrization $z = r\exp(\mathrm{i}\varphi)$, so that $\mathrm{d}^2z = r\,\mathrm{d}r\,\mathrm{d}\varphi$, where $r \in [0, R)$ and $\varphi \in [0, 2\pi[$, leads to the Stieljes moment problems

$$\forall n \in \mathbb{N}, \qquad \int_0^{R^2} \mathrm{d}u\, u^n h^{\pm}(u) = \frac{\left(K_{\pm}^0(\{n\})!\right)^2}{\left(\left(h_f^{\pm}(n-1)!\right)h_f^{\pm}(0)\right)^2}, \tag{66}$$

where $u = r^2$ and the functions $h^{\pm}(r^2)$ are

$$h^{+}(r^2) = \frac{4\pi^2}{3}|\mathcal{N}^{+}(r)|^2\mathcal{W}^{+}(r), \qquad h^{-}(r^2) = \frac{8\pi^2}{3}|\mathcal{N}^{-}(r)|^2\mathcal{W}^{-}(r). \tag{67}$$

Let us remark that the problems (66) have the same form of the Stieljes problems as was developed in [11]. Consequently, the same techniques as found therein could be used in order to determine a solvable and deformed resolution of the identity. Some explicit solutions will be furnished later. Provided these answers, we may assume that there exists a wide class of sets of $(k, \varepsilon, \kappa, f)$-VCSs fulfilling the Gazeau–Klauder axioms in a nonempty disk D_R of $\mathbb{C}$ distinguishable by specific choices of $K_{\pm}^0(\{n\}) > 0$ and $h_f^{\pm}(n)$, for instance.

4.2. Some expectation values and action-angle variables

We briefly give the expectation value of the Hamiltonian operator and discuss the atomic spin time evolution average.

The Hamiltonian mean value measured in any state is given by

$$\begin{aligned}\langle\mathcal{H}^{\mathrm{red}}\rangle = {}& |\mathcal{N}^{+}(|z|)|^2\cos^2\theta\sum_{n=0}^{\infty}\frac{|z|^{2n}}{\left(K_{+}^0(\{n\})!\right)^2}\left(\left(h_f^{+}(n-1)!\right)h_f^{+}(0)\right)^2 e_n^{+}\\ &+ |\mathcal{N}^{-}(|z|)|^2\sin^2\theta\sum_{n=0}^{\infty}\frac{|z|^{2n}}{\left(K_{-}^0(\{n\})!\right)^2}\left(\left(h_f^{-}(n-1)!\right)h_f^{-}(0)\right)^2 e_n^{-}.\end{aligned} \tag{68}$$

The average spin time evolution (atomic inversion in quantum optics) is defined by $\langle\sigma_3(t)\rangle = \langle U^{-1}(t)\sigma_3U(t)\rangle$, with the time evolution operator $U(t) = \exp(-\mathrm{i}\omega_0 t\mathcal{H}^{\mathrm{red}})$. We get a similar atomic inversion as in [11] with Rabi oscillation currently of the form

$$\begin{aligned}\Psi_n(t) &= \omega_0\left[(t+\tau_{+})e_n^{+} - (t+\tau_{-})e_n^{-}\right] + \phi - \varphi_{\lambda}(n)\\ &= \omega_0\Delta e_n t + \omega_0\left[\tau_{+}e_n^{+} - \tau_{-}e_n^{-}\right] + \phi - \varphi_{\lambda}(n)\end{aligned} \tag{69}$$

showing an explicit time dependence due to the mixed-spin matrix elements when $\lambda(\{N\}) \neq 0$ (in the limit $\lambda(\{N\}) \to 0$, oscillations collapse) and consisting only of Rabi oscillations as expected. This is a general property for quantum optics system prepared in a CS of the radiation field [18].

Klauder's argument [28] that the stability of CSs under time evolution could be translated by defining continuous parameters, so-called canonical action-angle variables (J, τ), should be recast for the purpose of VCSs, as defining canonically conjugate and continuous coordinates (J_ℓ, τ_ℓ), ℓ standing for the VCS index. Explicitly, if the Hamiltonian expectation value in any state can be written as

$$\langle \mathcal{H}^{\mathrm{red}} \rangle = J_+\omega_+ + J_-\omega_- = \sum_{\pm} J_\pm \omega_\pm, \tag{70}$$

where $\omega_\pm$ are some constant factors, we can identify through the action-angle variational principle

$$\int \mathrm{d}t \sum_{\pm} \left[\frac{\mathrm{d}\tau_\pm}{\mathrm{d}t} J_\pm - \omega_\pm J_\pm \right] \longleftrightarrow \int \mathrm{d}t \left[\left\langle \frac{\mathrm{i}}{\omega_0} \frac{\mathrm{d}}{\mathrm{d}t} \right\rangle - \langle \mathcal{H}^{\mathrm{red}} \rangle \right] \tag{71}$$

the following Hamiltonian equations:

$$\frac{\mathrm{d}\tau_\pm}{\mathrm{d}t} = \frac{\partial \langle \mathcal{H}^{\mathrm{red}} \rangle}{\partial J_\pm} = \omega_\pm, \qquad \frac{\mathrm{d}J_\pm}{\mathrm{d}t} = -\frac{\partial \langle \mathcal{H}^{\mathrm{red}} \rangle}{\partial \tau_\pm} = 0. \tag{72}$$

Then any shift of time $\tau_\pm \to \tau_\pm + t$ implies that $\omega_\pm = 1$ and, as a corollary, (72) involves the canonical action coordinates conjugated to $\tau_\pm$ of the form

$$\begin{aligned} J_+ &= |\mathcal{N}^+(|z|)|^2 \cos^2\theta \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\left(K_+^0(\{n\})!\right)^2} \left(\left(h_f^+(n-1)!\right) h_f^+(0)\right)^2 e_n^+, \\ J_- &= |\mathcal{N}^-(|z|)|^2 \sin^2\theta \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\left(K_-^0(\{n\})!\right)^2} \left(\left(h_f^-(n-1)!\right) h_f^-(0)\right)^2 e_n^-. \end{aligned} \tag{73}$$

4.3. Explicit solutions

Here, we sketch the way to obtain classes of solutions of the moment problems (66) either in canonical or in deformed situations.

A simple class of solutions. We note that the general algebraic restriction such that the ladder operators $\mathcal{M}^-$ and $\mathcal{M}^+$ obey a f-deformed oscillator algebra on $\mathcal{V}$, namely

$$\mathcal{M}^-\mathcal{M}^+ - \mathcal{M}^+\mathcal{M}^- = \sum_{n=0,\pm}^{\infty} \left|e_n^\pm\right\rangle(\{n+1\} - \{n\})\left\langle e_n^\pm\right|, \tag{74}$$

implies that $K_\pm^0(\{n\}) = \sqrt{\{n\}}$ with initial conditions $K_\pm^0(\{0\}) = 0$. The normalization series (61) prove to be as $\mathcal{N}^\pm(|z|) = \mathrm{e}^{-|z|^2/2}$ with infinite convergence radii if one further sets $\left(h_f^\pm(n)\right)^2 = (f(n+1))^2$. The subsequent moment problems (66) reduce to

$$\int_0^\infty \mathrm{d}u u^n h^\pm(u) = n!, \qquad n = 0, 1, 2, \ldots, \tag{75}$$

with solutions $h^\pm(u) = e^{-u}$ and associated weight factors as $\mathcal{W}^+(|z|) = 3/(4\pi^2)$ and $\mathcal{W}^-(|z|) = 3/(8\pi^2)$.

Hence, any family of $(k, \varepsilon, \kappa, f)$-VCSs possesses at least one exact solution to their resolution of identity. Canonical VCSs can be determined in the same way, with a constraint similar to (74), i.e. the ladder operator should obey an ordinary Fock–Heisenberg algebra.

A class of canonical solutions. The action identity constraint [28] also bears a class of solutions to the resolution of the identity equation only for the canonical case ($f(N) \to 1, \kappa \to 1$); a huge simplification occurring in this case. Consider the requirements

$$J_+ = \cos^2\theta\big(|z|^2 + e_0^+\big), \qquad J_- = \sin^2\theta\big(|z|^2 + e_0^-\big). \tag{76}$$

These statements govern the next relations, under a supplementary condition of a bounded from below energy spectrum such that $e_n^\pm - e_0^\pm \geqslant 0$,

$$K_\pm^0(n) = \sqrt{e_n^\pm - e_0^\pm}. \tag{77}$$

The spin–orbit decoupled model with $\lambda(N) = 0$ implies

$$\forall\, n \in \mathbb{N}, \qquad e_n^\pm - e_0^\pm = (1+\epsilon)n. \tag{78}$$

One concludes that $K_\pm^0(n) = \sqrt{(1+\epsilon)n}$ and $K_\pm^0(0) = 0$. The normalization factors of these nonlinear VCSs are $N^\pm(|z|)^{-2} = \exp[|z|^2/(1+\epsilon)]$ of infinite convergence radii. The moment problem related to the resolution of the identity of such states can be put in the form

$$\int_0^\infty \mathrm{d}u u^n h^\pm(u) = (1+\epsilon)^n n! \tag{79}$$

Changing variables as $u \to u/(1+\epsilon)$, it is not difficult to obtain $h^\pm(u) = \exp[-u/(1+\epsilon)]/(1+\epsilon)$ and to deduce the measure weight factors

$$\mathcal{W}^+(|z|) = \frac{3}{4\pi^2(1+\epsilon)}, \qquad \mathcal{W}^-(|z|) = \frac{3}{8\pi^2(1+\epsilon)}. \tag{80}$$

Solutions as (p,q) deformations. We can also find classes of NVCSs with exact solutions to their moment problem in $(p,q;\alpha,\beta,\ell)$-Burban deformed theory [25], in which case

$$f(N) = \sqrt{\frac{p^{-\alpha N-\beta} - q^{\alpha N+\beta}}{N(p^{-\ell} - q^{\ell})}}, \qquad h_f^\pm(N) = \left(\frac{q^\mu}{p^\nu}\right)^N \sqrt{l^\pm(p,q)} \tag{81}$$

with $0 < q < 1, p > 1, (pq)^\alpha < 1, \alpha \geqslant 0, \beta, \ell, \mu$ and ν being real parameters, and the positive real-valued functions $l^\pm(p,q)$ are such that $\lim_{(p,q)\to(1^+,1^-)} h_f^\pm(N) = 1$. Here, exact solutions to (66) can be expressed for $\beta = 0$ and by fixing the constant parameters (μ,ν,p,q,α) so that the convergence radii of norm series are nonvanishing. The problems (66) turn to $(p,q;\alpha,0,\ell)$-Ramanujan integrals of which solutions can be written as deformed generalized exponential (see [11] and a summary in the appendix). Nevertheless, there is a more general formulation recovering the $(p,q;\alpha,\beta,\ell)$-theory and still allowing the existence of a class solution that we propose to investigate. The multiparameter $(p,q;\alpha,\beta,\ell;\rho,\xi;\phi_1,\phi_2)$ quantum deformation is an extension of the $(p,q;\rho,\xi;\phi_1,\phi_2)$-deformation as settled in [26], introducing the new indices α, β and ℓ as found in (81). The deformation function in such a theory is

$$f(N) = \sqrt{\left(\frac{p^\rho}{q^\xi}\right)^N \frac{p^{-\alpha N-\beta}\phi_1(p,q) - q^{\alpha N+\beta}\phi_2(p,q)}{N(p^{-\ell} - q^{\ell})}}, \tag{82}$$

with the set of conditions over parameters

$$0 < \phi_1(p,q) < \phi_2(p,q), \qquad \frac{\phi_1(p,q)}{\phi_2(p,q)} = (pq)^{k_0}, \qquad k_0 \in \mathbb{N}, \tag{83}$$

$$(pq)^\alpha < 1, \qquad \alpha \geqslant 0, \qquad p > 1, \qquad 0 < q < 1, \qquad (\beta,\ell) \in \mathbb{R}^2, \tag{84}$$

which ensures the convergence of the upcoming infinite sums and products (see also [26] for relevant reductions). As a matter of continuity, $\phi_i(p,q)$, $i = 1, 2$, could be taken as continuous functions of the two parameters (p,q). As argued in [26], recovered for $(\alpha,\beta,\ell) = (1,0,1)$, the integer k_0 causes the existence of two ground states $(k_0 = 0, k_0)$ of the generalized harmonic oscillator Hamiltonian built in this framework.

Coming back to our problem, one imposes the next constraint on the algebra of ladder operators

$$\frac{q_0^{\xi}}{p_0^{\rho}}\mathcal{M}^-\mathcal{M}^+ - q_0^{\ell}\mathcal{M}^+\mathcal{M}^- = \sum_{n=0,\pm}^{\infty} |e_n^{\pm}\rangle \frac{p_0^{(\rho-\alpha)N-\beta}}{q_0^{\xi N}}\phi_1(p_0,q_0)\langle e_n^{\pm}| \tag{85}$$

$$\frac{q_0^{\xi}}{p_0^{\rho}}\mathcal{M}^-\mathcal{M}^+ - p_0^{-\ell}\mathcal{M}^+\mathcal{M}^- = \sum_{n=0,\pm}^{\infty} |e_n^{\pm}\rangle \frac{p_0^{\rho N}}{q_0^{(\xi-\alpha)N-\beta}}\phi_2(p_0,q_0)\langle e_n^{\pm}|, \tag{86}$$

with (p_0, q_0) being the new deformation parameters such that $p_0 > 1, 0 < q_0 < 1, (p_0q_0)^{\alpha} < 1$. A direct algebra shows that

$$\frac{q_0^{\xi}}{p_0^{\rho}}\left(K_{\pm}^0([n+1]_0)\right)^2 - q_0^{\ell}\left(K_{\pm}^0([n]_0)\right)^2 = \frac{p_0^{(\rho-\alpha)n-\beta}}{q_0^{\xi n}}\phi_1(p_0,q_0), \tag{87}$$

$$\frac{q_0^{\xi}}{p_0^{\rho}}\left(K_{\pm}^0([n+1]_0)\right)^2 - p_0^{-\ell}\left(K_{\pm}^0([n]_0)\right)^2 = \frac{p_0^{\rho n}}{q_0^{(\xi-\alpha)n-\beta}}\phi_2(p_0,q_0), \tag{88}$$

where

$$[n]_0 = [n]_{(p_0,q_0)} := \frac{p_0^{\rho n}}{q_0^{\xi n}}\frac{\left(p_0^{-\alpha n-\beta}\phi_1(p_0,q_0) - q_0^{\alpha n+\beta}\phi_2(p_0,q_0)\right)}{\left(p_0^{-\ell} - q_0^{\ell}\right)} \tag{89}$$

corresponds to the basic integer of the theory. The solutions to recurrence relations (87) and (88) are

$$K_{\pm}^0([n]_0) = \sqrt{[n]_{(p_0,q_0)}} \tag{90}$$

with initial values $K_{\pm}^0([0]_0) = [0]_0$. Note that the symmetry exchange $\left(p_0 \leftrightarrow q_0^{-1}\right)$, $(\phi_1 \leftrightarrow \phi_2)$ and $(\rho \leftrightarrow \xi)$ makes compatible the solutions of these recurrence relations. From the annihilation operator action, one should set $K_{\pm}^0([0]_0) = [0]_0 = 0$ and find a restriction through

$$p_0^{-\beta}\phi_1(p_0,q_0) - q_0^{\beta}\phi_2(p_0,q_0) = 0 \quad\Longleftrightarrow\quad \beta = k_0. \tag{91}$$

This latter relation indicates that the annihilation operator action on the ground state could vanish only when a Burban deformed theory coincides with a $(p, q; \rho, \xi; \phi_1, \phi_2)$-theory up to a deformation function, namely

$$F(N) = \sqrt{\left(\frac{p^{\rho}}{q^{\xi}}\right)^N p^{-\beta}\phi_1(p,q)\frac{(p^{-\ell} - q^{\ell})}{(p^{-1} - q^{1})}}. \tag{92}$$

Moreover, in the former study [11], the cancellation of $K_{\pm}^0([0]_0) = [0]_0 = 0$ could be realized only for a linear theory $\beta = 0$. At this stage, noting that $k_0 = \beta$ is still a free parameter, the below solutions prove to be more general than those obtained in [11].

The norm series of the NVCSs become, using the expression of $h_f^{\pm}(N)$ (81),

$$|\mathcal{N}^{\pm}(|z|)|^{-2} = \sum_{n=0}^{\infty}\frac{|z|^{2n}}{[n]_0!}\left(\frac{q^{\mu}}{p^{\nu}}\right)^{n(n-1)}(l^{\pm}(p,q))^n, \tag{93}$$

and have the radius of convergence

$$R_{\pm} = \lim_{n\to\infty}\left[\left(\frac{q^{\mu}}{p^{\nu}}\right)^{-2n}\frac{[n]_0}{l^{\pm}(p,q)}\right]^{1/2} \tag{94}$$

$$= \lim_{n\to\infty}\left[\left(p_0^{\rho-\alpha}p^{2\nu}q^{-2\mu}q_0^{-\xi}\right)^n p_0^{-\beta}\phi_1(p,q)\frac{1-(p_0q_0)^{\alpha n}}{l^{\pm}(p,q)\left(p_0^{-\ell}-q_0^{\ell}\right)}\right]^{1/2}. \tag{95}$$

Assuming that $p_0^{\rho-\alpha}p^{2\nu}q^{-2\mu}q_0^{-\xi} > 1$, $R_{\pm}$ are infinite. The moment problem (66) can be written as, for all $n \in \mathbb{N}$,

$$\int_0^{\infty} \mathrm{d}u\, u^n h^{\pm}(u) = \left(\frac{q^{\mu}}{p^{\nu}}\right)^{-n(n-1)}(l^{\pm}(p,q))^{-n}[n]_0!$$
$$= q_0^{-\xi\frac{n(n+1)}{2}}q^{-\mu n(n-1)}p_0^{\rho\frac{n(n+1)}{2}-\beta n}p^{-\nu n(n-1)}\Psi(p,q)^n[p^{\alpha},q^{\alpha};p^{\alpha},q^{\alpha}]_n, \tag{96}$$

$$\Psi(p,q) = \phi_1(p,q)\left(p_0^{-\ell}-q_0^{\ell}\right)^{-1}(l^{\pm}(p,q))^{-1}. \tag{97}$$

We now choose an appropriate set of parameters as

$$p_0 = p, \qquad q_0 = q, \qquad \frac{\xi}{2}+\mu = \frac{\alpha}{2}, \qquad \frac{\rho}{2}+\nu = 0, \tag{98}$$

so that $p_0^{\rho-\alpha}p^{2\nu}q^{-2\mu}q_0^{-\xi} = (pq)^{-\alpha} > 1$, implying infinite radii of convergence for the norm series. Afterward, making use of the (p^{α},q^{α})-extension of the Ramanujan integral, we derive the moment function (generalized exponential functions can be found in the appendix)

$$\begin{aligned} h^{\pm}(|z|^2) &= \Phi^{-1}(p,q)\frac{1}{\log(1/(pq)^{\alpha})}e_{(p^{\alpha},q^{\alpha})}(-|z|^2\Phi^{-1}(p,q)p^{-\alpha/2}), \\ \Phi(p,q) &= q^{\alpha-\xi}p^{-2\nu-\beta}\Psi(p,q). \end{aligned} \tag{99}$$

The norm series can be inferred as, $\left(K_{\pm}^{0}([n])\right)^2 = [n]$,

$$|\mathcal{N}^{\pm}(|z|)|^{-2} = \mathcal{E}^{(1/2,0)}_{(p^{\alpha},q^{\alpha})}(|z|^2q^{\xi-\alpha/2}p^{\beta+2\nu}\phi_1^{-1}(p,q)l^{\pm}(p,q)(p^{-\ell}-q^{\ell})), \tag{100}$$

and one can easily deduce the weight functions $\mathcal{W}^{\pm}(|z|)$ from (67). These weights generalize the measure as obtained in [11] with a freedom parametrized by the deformation functions $\phi_1(p,q)$, and the new extra parameters q^{ξ} and $p^{\beta+2\nu}$. In order to get previous results of [11], one has to set $\xi = 0 = \beta + 2\nu$ and $\phi_1(p,q) = 1$ in which case $\phi_2(p,q)$ corresponds to the monomial function $(pq)^{-\beta}$.

In summary, NVCSs associated with the nonlinear spin–orbit Hamiltonian are characterized by a unit vector of the sphere S^2 determined by coordinates θ and ϕ. Some appropriate constraints should be set in order that they could fulfill all the axioms of Gazeau–Klauder, namely continuity in the parameter $z \in \mathbb{C}$, temporal stability through a shift of the real parameters $\tau_{\pm} \to \tau_{\pm} + t$ and the overcompleteness property as a resolution of the Hilbert space $\mathcal{V}$. They distinguish from one another by different real positive factors $K_{\pm}^{0}(\{n\})$ parametrizing the freedom afforded by the annihilation operator action. An exact resolution of the unity over the total Hilbert space can be derived after specifying the remaining freedom.

5. Matrix formulation

The S^2 NVCSs have a natural extension as matrix NVCSs. In [11], we considered diagonal and quaternionic matrix domains. Here, we enlarge the study to normal (including diagonal complex matrices) and quaternionic matrix domains taking, of course, into account the new Hilbert space framework.

To proceed, we write the diagonal matrix annihilation operator associated with (41) as

$$\mathbb{M}^- = \sum_{n=0,\pm}^{\infty} |n-1\rangle\langle n| \otimes K(\{n\})|\pm\rangle\langle\pm|, \qquad K(\{n\}) = \mathrm{diag}(K_+(\{n\}), K_-(\{n\})), \tag{101}$$

where we assume that $K(\{n\})$ is in diagonal form. By similarity, we project the annihilation operator onto the basis $|e_n^\pm\rangle$ as

$$\mathcal{M}^- = \mathcal{U}\mathbb{M}^-\mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} K(\{n\})\big|e_{n-1}^\pm\big\rangle\big\langle e_n^\pm\big|. \tag{102}$$

The same eigenvalue problem as in (29) is found as

$$\mathcal{M}^-|(z,w);\tau_\pm;\pm\rangle = \widetilde{\mathfrak{Z}}(z,w)\widetilde{\mathbb{Q}}_{\mathcal{V}}|(z,w);\tau_\pm;\pm\rangle, \tag{103}$$

with the same shape for operators $\widetilde{\mathfrak{Z}}(z,w)$ and $\widetilde{\mathbb{Q}}_{\mathcal{V}}$. Still assuming that $\mathfrak{Z} = \mathcal{U}^\dagger\widetilde{\mathfrak{Z}}\mathcal{U}$ is a complex constant matrix, we define $\mathbb{Q}_{\mathcal{V}} := \mathcal{U}^\dagger\widetilde{\mathbb{Q}}_{\mathcal{V}}\mathcal{U}$, so that (103) can be put into the form

$$\mathbb{M}^-|\mathfrak{Z};\tau_\pm;\pm\rangle = \mathfrak{Z}\mathbb{Q}_{\mathcal{V}}|\mathfrak{Z};\tau_\pm;\pm\rangle, \qquad |\mathfrak{Z};\tau_\pm;\pm\rangle = \mathcal{U}^\dagger|(z,w);\tau_\pm;\pm\rangle. \tag{104}$$

Let us assume that $\mathbb{Q}_{\mathcal{V}}$ admits the expansion

$$\mathbb{Q}_{\mathcal{V}}^{\mathfrak{U}} = \sum_{n=0,\pm}^{\infty} |n\rangle\langle n| \otimes \mathfrak{U}h_f(n)\mathfrak{U}^\dagger|\pm\rangle\langle\pm|, \qquad h_f(n) = \mathrm{diag}\big(h_f^+(n), h_f^-(n)\big),$$

with $\mathfrak{U}$ being an element of the unitary group $U(2)$ or an element of $SU(2)$. Let us fix $\mathfrak{Z}$ as a normal matrix and $\mathfrak{Z}_{\mathrm{quat}}$ as a quaternionic matrix, i.e., they satisfy

$$\mathfrak{Z}^\dagger\mathfrak{Z} = \mathfrak{Z}\mathfrak{Z}^\dagger, \qquad \mathfrak{Z} = V\mathrm{diag}(z,w)V^\dagger, \qquad V \in U(2), \tag{105}$$

$$\mathfrak{Z}_{\mathrm{quat}} = U\mathrm{diag}(z,\bar{z})U^\dagger, \qquad U \in SU(2), \tag{106}$$

$$U = u_{\phi_1}u_\theta u_{\phi_2}, \qquad u_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & \mathrm{i}\sin\frac{\theta}{2} \\ \mathrm{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \qquad u_{\phi_i} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\phi_i}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\phi_i}{2}} \end{pmatrix}, \tag{107}$$

for $\theta \in [0,\pi]$ and $\phi_i \in]0, 2\pi]$. Therefore, the operators

$$\mathfrak{Z}_f := \mathfrak{Z}\mathbb{Q}_{\mathcal{V}}^V = \sum_{n=0,\pm}^{\infty} |n\rangle\langle n| \otimes V\mathfrak{Z}_{\mathrm{diag}}h_f(n)V^\dagger|\pm\rangle\langle\pm|, \tag{108}$$

$$\mathfrak{Z}_{\mathrm{diag}} := \mathrm{diag}(z,w), \tag{109}$$

$$\mathfrak{Z}_{\mathrm{quat},f} := \mathfrak{Z}_{\mathrm{quat}}\mathbb{Q}_{\mathcal{V}}^U = \sum_{n=0,\pm}^{\infty} |n\rangle\langle n| \otimes U\mathfrak{Z}_{\mathrm{qdiag}}h_f(n)U^\dagger|\pm\rangle\langle\pm|, \tag{110}$$

$$\mathfrak{Z}_{\mathrm{qdiag}} := \mathrm{diag}(z,\bar{z}) \tag{111}$$

satisfy the equations

$$\mathbb{M}_V^-|\mathfrak{Z};\tau_\pm;\pm\rangle = \mathfrak{Z}_f|\mathfrak{Z};\tau_\pm;\pm\rangle, \tag{112}$$

$$\mathbb{M}_U^-|\mathfrak{Z}_{\mathrm{quat}};\tau_\pm;\pm\rangle = \mathfrak{Z}_{\mathrm{quat},f}|\mathfrak{Z}_{\mathrm{quat}};\tau_\pm;\pm\rangle, \tag{113}$$

obtained by substituting in (104), $\mathbb{Q}_{\mathcal{V}}$ by $\mathbb{Q}_{\mathcal{V}}^{\mathfrak{U}}$, for $\mathfrak{U} = V, U$, and using the new group dependent annihilation operators

$$\mathbb{M}_{\mathfrak{U}}^{-} = \sum_{n=0,\pm}^{\infty} |n-1\rangle\langle n| \otimes \mathfrak{U}K(\{n\})\mathfrak{U}^{\dagger}|\pm\rangle\langle\pm|, \qquad \mathfrak{U} = V, U. \tag{114}$$

The procedure as applied in section 4.1 can be used for obtaining the general solutions of the eigenvalue problem (112) and (113) with all Gazeau–Klauder properties. We have the general form of Gazeau–Klauder NVCSs:

- For normal matrices

$$|\mathfrak{Z};\tau_{\pm};\pm\rangle = N(\mathfrak{Z})\sum_{n=0}^{\infty}|n\rangle \otimes V(K(\{n\})!)^{-1}V^{\dagger}\mathfrak{Z}_{f}^{n}|\pm\rangle \tag{115}$$

$$= N(\mathfrak{Z})\sum_{n=0}^{\infty}|n\rangle \otimes V R_0(n)\exp[-\mathrm{i}\omega_0\tau e_n]\mathfrak{Z}_{\mathrm{diag}}^{n}V^{\dagger}|\pm\rangle, \tag{116}$$

$$N(\mathfrak{Z})^{-2} = \sum_{n=0}^{\infty}\mathrm{Tr}[V|R_0(n)|^2\,\mathrm{diag}(|z|^{2n}, |w|^{2n})V^{\dagger}]$$

$$= \sum_{n=0}^{\infty}\left(|z|^{2n}|R_{+}^{0}(n)|^2 + |w|^{2n}|R_{-}^{0}(n)|^2\right), \tag{117}$$

$$R^0(n) = \mathrm{diag}\left(R_{+}^{0}(n), R_{-}^{0}(n)\right) = (K^0(\{n\})!)^{-1}(h_f(n-1)!h_f(0)), \tag{118}$$

where $K^0(\{n\}) = \mathrm{diag}\left(K_{+}^{0}(\{n\}), K_{-}^{0}(\{n\})\right)$, $K_{\pm}^{0}(\{n\}) = |K_{\pm}(\{n\})|$, $N(\mathfrak{Z})$ is the normalization factor, $\tau = \mathrm{diag}(\tau_{+}, \tau_{-})$ and $e_n = \mathrm{diag}\left(e_n^{+}, e_n^{-}\right)$. Note that the convergence radii of series (117) are such that $|z| \leqslant L_{+}, |w| \leqslant L_{-}$ and $L_{\pm} = \lim_{n\to\infty} K_{\pm}^{0}(\{n\})/|h_{\pm f}(n-1)|$.

- For quaternion matrices

$$|\mathfrak{Z}_{\mathrm{quat}};\tau_{\pm};\pm\rangle = N(\mathfrak{Z}_{\mathrm{quat}})\sum_{n=0}^{\infty}|n\rangle \otimes U(K(\{n\})!)^{-1}U^{\dagger}\mathfrak{Z}_{\mathrm{quat},f}^{n}|\pm\rangle \tag{119}$$

$$= N(\mathfrak{Z}_{\mathrm{quat}})\sum_{n=0}^{\infty}|n\rangle \otimes U R_0(n)\exp[-\mathrm{i}\omega_0\tau e_n]\mathfrak{Z}_{\mathrm{qdiag}}^{n}U^{\dagger}|\pm\rangle, \tag{120}$$

$$N(\mathfrak{Z}_{\mathrm{quat}})^{-2} = \sum_{n=0}^{\infty}\mathrm{Tr}[U|R_0(n)|^2|z|^{2n}\mathbb{I}_2U^{\dagger}]$$

$$= \sum_{n=0}^{\infty}|z|^{2n}\left(\left|R_{+}^{0}(n)\right|^2 + \left|R_{-}^{0}(n)\right|^2\right), \tag{121}$$

with the norm series convergence radius

$$L = \lim_{n\to\infty}\left[\frac{\left|R_{+}^{0}(n)\right|^2 + \left|R_{-}^{0}(n)\right|^2}{\left|R_{+}^{0}(n+1)\right|^2 + \left|R_{-}^{0}(n+1)\right|^2}\right]^{\frac{1}{2}}. \tag{122}$$

The NVCSs (116) and (120) are continuous, normalized according to the definition

$$\sum_{\pm}\langle\mathcal{Z};\tau_{\pm};\pm|\mathcal{Z};\tau_{\pm};\pm\rangle = 1, \qquad \mathcal{Z} = \mathfrak{Z}, \quad \mathfrak{Z}_{\mathrm{quat}}, \tag{123}$$

and stable under the time evolution operator

$$U^{\mathfrak{U}}(t) = \exp[-\mathrm{i}\omega_0 t \mathfrak{U}\mathbb{H}^{\mathrm{red}}\mathfrak{U}^\dagger] = \mathfrak{U}\exp[-\mathrm{i}\omega_0 t \mathbb{H}^{\mathrm{red}}]\mathfrak{U}^\dagger, \qquad \mathfrak{U} = V, U, \tag{124}$$

i.e.

$$U^{\mathfrak{U}}(t)|\mathcal{Z}; \tau_\pm; \pm\rangle = |\mathcal{Z}; \tau_\pm + t; \pm\rangle. \tag{125}$$

The action identity axiom could also be inferred by assigning the action variables to

$$J^{\mathfrak{U}}_\pm = \langle \mathcal{Z}; \tau_\pm; \pm|\mathfrak{U}\mathbb{H}^{\mathrm{red}}\mathfrak{U}^\dagger|\mathcal{Z}; \tau_\pm; \pm\rangle, \qquad \mathfrak{U} = V, U. \tag{126}$$

To recover the original NVCSs over the basis $|e^\pm_n\rangle$, we derive the state (104) as follows:

$$|(z, w); \tau_\pm; \pm\rangle_{\mathcal{Z}} = \mathfrak{U}\mathcal{U}\mathfrak{U}^\dagger|\mathcal{Z}; \tau_\pm; \pm\rangle \tag{127}$$

$$= N(\mathcal{Z}) \sum_{n=0}^{\infty} \mathfrak{U}|e^\pm_n\rangle R^\pm_0(n) \exp\left[-\mathrm{i}\omega_0\tau_\pm e^\pm_n\right]\mathcal{Z}^n_\pm \mathfrak{U}^\dagger_\pm, \tag{128}$$

with $\mathcal{Z}_\pm = \langle\pm|\mathcal{Z}_{\mathrm{diag}}|\pm\rangle$ and $\mathfrak{U}^\dagger_\pm = \langle\pm|\mathfrak{U}^\dagger|\pm\rangle$, so that the operator $\widetilde{\mathfrak{Z}}$ has the form

$$\widetilde{\mathfrak{Z}}_{\mathcal{Z}} = \sum_{n=0,\pm}^{\infty} \mathfrak{U}|e^\pm_n\rangle\mathcal{Z}_\pm\langle e^\pm_n|\mathfrak{U}^\dagger. \tag{129}$$

One notes that the normalization factor of $|(z, w); \tau_\pm; \pm\rangle$ remains the same as that of $|\mathcal{Z}; \tau_\pm; \pm\rangle$ as expected from any unitary transformation.

First, let us treat the resolution of the identity related to the normal matrix domain. The overcompleteness relation of the normal matrix NVCSs is given by

$$\mathbb{I}_{\mathcal{V}} = \sum_{\pm} \int_{\mathcal{D}} \mathrm{d}\mu(\mathfrak{Z})|\mathfrak{Z}; \tau_\pm; \pm\rangle\langle\mathfrak{Z}; \tau_\pm; \pm|, \tag{130}$$

where the domain $\mathcal{D}$ and the measure $\mathrm{d}\mu(\mathfrak{Z})$ are to be defined. Considering the parametrization of the variable $\mathfrak{Z} = V\mathrm{diag}(z, w)V^\dagger$ as

$$z = r_+\,\mathrm{e}^{\mathrm{i}\theta_+}, \qquad w = r_-\,\mathrm{e}^{\mathrm{i}\theta_-}, \qquad r_\pm \in [0, L_\pm), \qquad \theta_\pm \in [0, 2\pi[, \tag{131}$$

the domain of integration $\mathcal{D}$ is given by

$$\mathcal{D} = [0, L_+) \times [0, L_-) \times \{[0, 2\pi[\}^2 \times U(2), \tag{132}$$

where one should include the Lie group $U(2)$. Therefore, the measure $\mathrm{d}\mu(\mathfrak{Z})$ is of the form

$$\mathrm{d}\mu(\mathfrak{Z}) = N(\mathfrak{Z})^{-2}\mathcal{W}_+(r_+)\mathcal{W}_-(r_-)r_+r_-\,\mathrm{d}r_+\,\mathrm{d}r_-\,\mathrm{d}\theta_+\,\mathrm{d}\theta_-\,\mathrm{d}\Omega_{U(2)}(V). \tag{133}$$

Here $\mathrm{d}\Omega_{U(2)}(V)$ is the invariant Haar measure over $U(2)$ normalized to 1 and $\mathcal{W}_\pm(r_\pm)$ are weight factors to be fixed later.

After integration over the angle variables $\theta_\pm$, the identity (130) involves the next integral over the group $U(2)$,

$$\int_{U(2)} \mathrm{d}\Omega_{U(2)}(V)V\left(\left(R^0_+(n)\right)^2 r_+^{2n}|+\rangle\langle+| + \left(R^0_-(n)\right)^2 r_-^{2n}|-\rangle\langle-|\right)V^\dagger$$

$$= \frac{1}{2}r_+^{2n}\left(R^0(n)_+\right)^2 + \frac{1}{2}r_-^{2n}\left(R^0(n)_-\right)^2, \tag{134}$$

where we have used the orthogonality condition on a compact group [1]

$$\int_{U(2)} \mathrm{d}\Omega_{U(2)}(V)V|\pm\rangle\langle\pm|V^\dagger = \frac{1}{2}\mathbb{I}_2. \tag{135}$$

Here $\{|\pm\rangle\}$ plays the role of the canonical basis of $\mathbb{C}^2$. We end with the moment problems

$$\int_0^{L_\pm^2} \mathrm{d}u_\pm\, u_\pm^n h_\pm(u_\pm) = \left(K_\pm^0(\{n\})!\right)^2 / (h_{\pm f}(n-1)! h_{\pm f}(0))^2, \tag{136}$$

where $u_\pm = r_\pm^2$ and the functions $h_\pm(u_\pm) = \pi \mathcal{W}_\pm(u_\pm)$.

Solution to (136) could be found along the lines of section 4.3 for canonical and deformed situations. We give here the simplest solution obtained by stressing the ladder operators to satisfy $[\mathbb{M}^-, \mathbb{M}^+] = (\{N+1\} - \{N\})\mathbb{I}_2$, implying $K_\pm(\{n\}) = \{n\}$ and fixing $(h_{\pm f}(n))^2 = (f(n+1))^2$. The norm series (117) are $N(\mathfrak{Z})^{-2} = \left(e^{r_+^2} + e^{r_-^2}\right)$ such that $L_\pm = \infty$. One can solve (136) and find $h_\pm(u_\pm) = e^{-u_\pm}$ from which the weight factors $\mathcal{W}_\pm(u_\pm)$ and the measure

$$\mathrm{d}\mu(\mathfrak{Z}) = \frac{1}{\pi^2}\left(e^{-r_+^2} + e^{-r_-^2}\right) r_+ r_- \mathrm{d}r_+ \mathrm{d}r_-\, \mathrm{d}\theta_+\, \mathrm{d}\theta_-\, \mathrm{d}\Omega_{U(2)}(V) \tag{137}$$

are deduced. For other deformed theories, one can also show that these moment problems find solutions using appropriate deformed exponential functions. Finally, the correct measure for the anterior $\widetilde{\mathfrak{Z}}$-NVCSs can easily be obtained by using the fact that $\mathrm{d}\mu(\mathfrak{Z})$ is an invariant measure.

Concerning the quaternionic matrix domain, a similar treatment can be applied. However, we use a different technique in order to avoid the integration over the Lie group. The measure will be endowed with the following parametrization of the quaternions $\mathfrak{Z}_{\text{quat}} = U\, \mathrm{diag}(z, \bar{z}) U^\dagger$ as, using direct expansion of (106),

$$\mathfrak{Z}_{\text{quat}} = r(\cos\xi \mathbb{I}_2 + \mathrm{i}\sin\xi\sigma), \qquad \sigma = \begin{pmatrix} \cos\theta & \mathrm{e}^{\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{-\mathrm{i}\phi}\sin\theta & -\cos\theta \end{pmatrix}, \tag{138}$$

where

$$z = r\,\mathrm{e}^{\mathrm{i}\xi}, \qquad r \in [0, L), \quad \xi \in [0, 2\pi[, \quad \theta \in [0, \pi[, \quad \phi \in [0, 2\pi[. \tag{139}$$

Note that the Lie group $SU(2)$ dependence has been traded for a S^2 unit vector indices. It is then crucial to observe that, since $\sigma^2 = \mathbb{I}_2$,

$$\mathfrak{Z}_{\text{quat}} = r \exp[\mathrm{i}\xi\sigma], \tag{140}$$

thence any power of $\mathfrak{Z}_{\text{quat}}$ can easily be deduced and $(\mathfrak{Z}_{\text{quat}})^\dagger = r\exp[-\mathrm{i}\xi\sigma]$.

The domain of integration $\mathcal{D}$ is nothing but

$$\mathcal{D} = [0, L) \times [0, 2\pi[\times S^2 \tag{141}$$

for which the measure $\mathrm{d}\mu(\mathfrak{Z}_{\text{quat}})$ appears as

$$\mathrm{d}\mu(\mathfrak{Z}_{\text{quat}}) = N(\mathfrak{Z})^{-2}\mathcal{W}(r) r\, \mathrm{d}r\, \mathrm{d}\xi\, \mathrm{d}\mu_{S^2}(\theta, \phi), \qquad \mathrm{d}\mu_{S^2}(\theta,\phi) = \frac{1}{4\pi}\sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi \tag{142}$$

with $\mathcal{W}(r)$ being a weight factor to be determined. A straightforward algebra induces the moment problem

$$\int_0^{L^2} \mathrm{d}u\, u^n h(u) = \left(K_\pm^0(\{n\})!\right)^2 / (h_{\pm f}(n-1)! h_{\pm f}(0))^2, \tag{143}$$

where $u = r^2$ and the functions $h(u) = \pi\mathcal{W}(u)$. Again, we can single out a solution of the problem (143) by constraining the function $\left(h_{+f}^0(n)\right)^2 = \left(h_{-f}^0(n)\right)^2 = (f(n+1))^2$, and the algebra to be such that $[\mathbb{M}^-, \mathbb{M}^+] = (\{N+1\} - \{N\})\mathbb{I}_2$ implying $K_\pm(\{n\}) = \{n\}$. Indeed, we find that the norm series (121) is $N(\mathfrak{Z}_{\text{quat}})^{-2} = \left(2e^{r^2}\right)$ with convergence radius $L = \infty$. Therefore, the solution of (143) as $h(u) = e^{-u}$ corresponds to a measure

$$\mathrm{d}\mu(\mathfrak{Z}_{\text{quat}}) = \frac{1}{2\pi^2} r\, \mathrm{d}r\, \mathrm{d}\xi\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}\mu_{S^2}. \tag{144}$$

Measures (137) and (144) have been discussed in [11, 33]. Finally, the same comments about solvable deformed theories as over normal NVCSs remain true also for quaternion NVCSs.

6. Deformed displacement operators, dual states and *T* operators

Unitary displacement operators for VCSs over the matrix domain have a well-defined sense via the tensor product of matrix and Hilbert spaces [1]. Here, we need to define deformed versions of these operators that still generate the S^2 or matrix NVCSs. A step forward is to consider the displacement operators in the context of NCSs [29] which are mainly exploited in deformed quantum optics [5–7]. In [9], deformed inverse bosonic operators are used and, from these, their dual NCSs are defined. The purpose is to extend these operators to deformed versions of displacement operators and to investigate how they generate S^2 and matrix NVCSs.

6.1. S^2-displacement operators

Consider the annihilation operator

$$\mathcal{M}^- = \sum_{n=0,\pm}^{\infty} \left|e_{n-1}^{\pm}\right\rangle K_{\pm}(\{n\}) \left\langle e_n^{\pm}\right|, \tag{145}$$

where $K_{\pm}(\{n\})$ are free functions of $\{n\}$. Let us define a new operator

$$\mathcal{B}^+ = \sum_{n=0,\pm}^{\infty} \left|e_{n+1}^{\pm}\right\rangle \overline{G_{\pm}(\{n+1\})} \left\langle e_n^{\pm}\right|, \tag{146}$$

where $\overline{G_{\pm}(\{n\})}$ are new functions of number $\{n\}$ and impose the condition

$$[\mathcal{M}^-, \mathcal{B}^+] = \mathbb{I}_{\mathcal{V}}. \tag{147}$$

Therefore, a direct evaluation of (147) proves that the pair $(K_{\pm}(\{n\}), \overline{G_{\pm}(\{n\})})$ should satisfy

$$K_{\pm}(\{1\})\overline{G_{\pm}(\{1\})} = 1, \tag{148}$$

$$K_{\pm}(\{p+1\})\overline{G_{\pm}(\{p+1\})} - K_{\pm}(\{p\})\overline{G_{\pm}(\{p\})} = 1, \qquad \forall\, p \geqslant 1. \tag{149}$$

A simple solution to this problem is

$$G_{\pm}(\{0\}) = 0, \qquad \overline{G_{\pm}(\{p\})} = p(K_{\pm}(\{p\}))^{-1}, \qquad \forall p \geqslant 1. \tag{150}$$

Moreover, one can check that

$$\left[\widetilde{\mathbb{Q}}_{\mathcal{V}}^{-1}\mathcal{M}^-, \mathcal{B}^+\widetilde{\mathbb{Q}}_{\mathcal{V}}\right] = \mathbb{I}_{\mathcal{V}}, \qquad \widetilde{\mathbb{Q}}_{\mathcal{V}}^{-1} = \sum_{n=0,\pm}^{\infty} \left|e_n^{\pm}\right\rangle \left(h_f^{\pm}(n)\right)^{-1} \left\langle e_n^{\pm}\right|. \tag{151}$$

We are immediately in a position to define a displacement operator for S^2 NVCS theory by introducing

$$\mathcal{D}_f = \mathrm{e}^{z\mathcal{B}^+\widetilde{\mathbb{Q}}_{\mathcal{V}} - \bar{z}\widetilde{\mathbb{Q}}_{\mathcal{V}}^{-1}\mathcal{M}^-} \tag{152}$$

and the NVCSs can be rebuilt from the action of operator $\mathcal{D}_f$ onto the ground states as

$$|z; \tau_{\pm}; \theta, \phi\rangle = \mathcal{D}_f\left\{\mathrm{e}^{\frac{1}{2}|z|^2}\left[\mathcal{N}^+(|z|)\cos\theta\, \mathrm{e}^{-\mathrm{i}\omega\tau_+ e_0^+}\left|e_0^+\right\rangle + \mathcal{N}^-(|z|)\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta\, \mathrm{e}^{-\mathrm{i}\omega\tau_- e_0^-}\left|e_0^-\right\rangle\right]\right\}. \tag{153}$$

Of course, the limit $f(N) \to 1$ implies that $\mathcal{D}_f$ converges to a kind of ordinary displacement operator $\mathrm{e}^{za^\dagger - \bar{z}a}$ of usual CSs recovered for $K(n) = \sqrt{n}$.

Another class of NCVSs, so-called 'dual' NVCSs, could be introduced by noting the fact that

$$[\mathcal{M}^+, \mathcal{B}^-] = \mathbb{I}_{\mathcal{V}}, \qquad \left[\widetilde{\mathbb{Q}}_{\mathcal{V}}\mathcal{B}^-, \mathcal{M}^+\widetilde{\mathbb{Q}}_{\mathcal{V}}^{-1}\right] = \mathbb{I}_{\mathcal{V}}, \qquad \mathcal{B}^- = (\mathcal{B}^+)^\dagger \tag{154}$$

and by defining a new displacement operator of the form

$$\mathcal{D}'_f = \mathrm{e}^{z\mathcal{M}^+\widetilde{\mathbb{Q}}_{\mathcal{V}}^{-1} - \bar{z}\widetilde{\mathbb{Q}}_{\mathcal{V}}\mathcal{B}^-}. \tag{155}$$

Applying the latter on ground states, we get the set of dual NVCSs as

$$|z;\tau_\pm;\theta,\phi\rangle' = \mathcal{D}'_f\left\{\mathrm{e}^{\frac{1}{2}|z|^2}\left[\mathcal{N}'^+(|z|)\cos\theta\,\mathrm{e}^{+\mathrm{i}\omega\tau_+e_0^+}\left|e_0^+\right\rangle + \mathcal{N}'^-(|z|)\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta\,\mathrm{e}^{+\mathrm{i}\omega\tau_-e_0^-}\left|e_0^-\right\rangle\right]\right\} \tag{156}$$

where one needs to introduce new normalization factors

$$\mathcal{N}'^\pm(|z|) = \left[\sum_{n=0}^{\infty}\frac{|z|^{2n}}{(n!)^2}\frac{\left(K_\pm^0(\{n\})!\right)^2}{\left((h_f^\pm(n-1)!)h_f^\pm(0)\right)^2}\right]^{-1/2} \tag{157}$$

of convergence radii $R'_\pm$

$$R'_\pm = \lim_{n\to\infty}\left[\frac{nh_f^\pm(n-1)}{K_\pm^0(\{n\})}\right]. \tag{158}$$

Noting the change of phase $+\mathrm{i}\omega\tau_\pm e_0^\pm$ (156), a peculiar notion of evolution is encoded in the dual state definition. We will discuss this below according to the matrix formulation.

One could ask if the NVCSs and their analog dual states could be simultaneously solvable in the sense that they give an exact resolution of the identity. Let us consider the specific instance when $K_\pm^0(\{n\}) = \sqrt{\{n\}}$ and $h_f^\pm(n) = sf(n+1)$, $s = \pm$, since this case proves to be solvable for NVCSs (see section 4.3). Therefore, the resolution of the identity for dual NVCSs reduces to the moment problem (75), for $s = +$, $R' = \lim_{n\to\infty}\sqrt{n} = \infty$ and then getting $h_\pm(u_\pm) = e^{-u_\pm}$. These solutions consist of those of any set of CSs coinciding with their dual counterpart in the absence of deformation, i.e., $f(N) \to 1$ [7]. For other deformed theories, any answer could be given without making a careful analysis. However, disregarding the prime NVCSs, we can always find an exact resolution of the moment problem for (p,q) deformation of the dual NVCSs. Indeed, one has to switch the role played by $K(\{n\})$ and $h_f^\pm(n-1)$ along the lines of the resolution in section 4.3 and then totally finds similar solutions. Another remarkable feature introduced by operator $\mathbb{Q}_{\mathcal{V}}$ is the large number of possible displacement operators. We note that, since on the one hand $\mathbb{Q}_{\mathcal{V}}$ and $\mathcal{M}^\pm$, and on the other hand $\mathbb{Q}_{\mathcal{V}}$ and $\mathcal{B}^\pm$ are noncommuting operators, it becomes possible to define, by any order of composition, the displacement operators. Adding now $\mathbb{Q}_{\mathcal{V}}^{-1}$ into the game, the set of significant operators is even more large (for instance, note that $\left[\mathcal{B}^-\widetilde{\mathbb{Q}}_{\mathcal{V}}, \widetilde{\mathbb{Q}}_{\mathcal{V}}^{-1}\mathcal{M}^+\right] = \mathbb{I}_{\mathcal{V}}$). The corresponding displacement operators obviously generate different sets of NVCSs for a particular order and by coupling $\mathbb{Q}_{\mathcal{V}}$ or $\mathbb{Q}_{\mathcal{V}}^{-1}$ to $\mathcal{M}^\pm$ or $\mathcal{B}^\pm$. To study all these NVCSs is, of course, an interesting issue that we will postpone to a forthcoming work consisting in the classification of these families using different criteria (for instance, exact solution of their moment problem and which of them are simultaneously solvable, or by sharper analytical properties, etc). Finally, one remarks that, in deformed theory, an initial family of NVCSs could have many solvable 'dual' counterparts.

6.2. Matrix displacement operators

The same notion of deformed displacement operators also makes sense in a matrix theory. In this paragraph, we only focus on normal matrix NVCSs; the case of quaternionic domain could easily be inferred. Consider the matrix annihilation operator as given by (101) and the operator (same notations as in section 5 are used)

$$\mathbb{B}_V^+ = \sum_{n=0,\pm}^{\infty}|n+1\rangle\langle n|\otimes V\overline{G(\{n+1\})}V^\dagger|\pm\rangle\langle\pm|, \qquad V\in U(2), \tag{159}$$

such that $\overline{G(\{n\})} = \mathrm{diag}(\overline{G_+(\{n\}}, \overline{G_-(\{n\})})$. Then, the following algebra is satisfied

$$\left[\mathbb{M}_V^-, \mathbb{B}_V^+\right] = \mathbb{I}_{\mathcal{V}}, \tag{160}$$

if and only if, at the matrix level

$$K(\{1\})\overline{G(\{1\})} = \mathbb{I}_2, \tag{161}$$

$$K(\{p+1\})\overline{G(\{p+1\})} - K(\{p\})\overline{G(\{p\})} = \mathbb{I}_2, \qquad \forall p \geqslant 1. \tag{162}$$

These problems have the solutions

$$\overline{G(\{0\})} = 0, \qquad \overline{G(\{p\})} = p(K(\{p\}))^{-1}, \qquad \forall p \geqslant 1. \tag{163}$$

Consequently, the matrix operator,

$$\mathbb{D}_f = \exp\left[\mathbb{B}_V^+ \cdot \mathfrak{Z}_f - \mathfrak{Z}_f^+ \cdot \mathbb{M}_V^-\right], \tag{164}$$

$$\mathfrak{Z}_f^+ := \left(\mathbb{Q}_{\mathcal{V}}^V\right)^{-1}\mathfrak{Z}^\dagger = \sum_{n=0,\pm}^{\infty} |n\rangle\langle n| \otimes V\mathfrak{Z}_{\mathrm{diag}}^\dagger(h_f(n))^{-1}V^\dagger|\pm\rangle\langle\pm|, \tag{165}$$

defines a displacement operator for normal matrix NVCSs. Hence, we have

$$|\mathfrak{Z};\tau_\pm;\pm\rangle = \mathbb{D}_f\left[\exp\left[\tfrac{1}{2}\mathfrak{Z}^\dagger\mathfrak{Z}\right]N(\mathfrak{Z})V\exp[-\mathrm{i}\omega_0\tau e_0]V^\dagger|0,\pm\rangle\right], \tag{166}$$

where $N(\mathfrak{Z})$ is the normalization factor given by (117) and the phase factor is introduced to maintain the theory stable under the time evolution along the lines of section 4.1. We should comment that one should more rigorously write the exponent of the exponential factor as written in (166) as follows:

$$\mathfrak{Z}^\dagger\mathfrak{Z} = \sum_{n,\pm}^{\infty} |n\rangle\langle n| \otimes \mathfrak{Z}^\dagger\mathfrak{Z}|\pm\rangle\langle\pm|. \tag{167}$$

The duals of matrix displacement operators also have a well-defined sense. Indeed, defining $\mathbb{B}_V^- = \left(\mathbb{B}_V^+\right)^\dagger$, the algebra

$$\left[\mathbb{B}_V^-, \mathbb{M}_V^+\right] = \mathbb{I}_{\mathcal{V}} \tag{168}$$

is trivially satisfied. We introduce the deformed dual operator

$$\mathbb{D}'_f = \exp\left[\mathbb{M}^+ \cdot \mathfrak{Z}_f - \mathfrak{Z}_f^+ \cdot \mathbb{B}^-\right] \tag{169}$$

and deduce dual matrix NVCSs as

$$\begin{aligned}|\mathfrak{Z};\tau_\pm;\pm\rangle' &= \mathbb{D}'_f\left[\exp\left[\frac{1}{2}\mathfrak{Z}^+\mathfrak{Z}\right]N'(\mathfrak{Z})\exp[+\mathrm{i}\omega_0\tau e_0]|0,\pm\rangle\right]\\ &= N'(\mathfrak{Z})\sum_{n=0}^{\infty}|n\rangle\otimes V\frac{R'^0(n)}{n!}\exp[+\mathrm{i}\omega_0\tau e_n]\mathfrak{Z}_{\mathrm{diag}}^n V^\dagger|\pm\rangle,\end{aligned} \tag{170}$$

$$N'(\mathfrak{Z}) = \sum_{n=0}^{\infty}\left[\frac{|z|^{2n}}{(n!)^2}\left(R_+'^0(n)\right)^2 + \frac{|w|^{2n\,2}}{(n!)}\left(R_-'^0(n)\right)^2\right], \tag{171}$$

where $R'^0(n) = K(\{n\})!\left(h_f^0(n-1)!h_f^0(0)\right)$. A notable feature of dual NVCSs is the change of sign of the phase factor in (170). This is related to the fact that if we keep the same notation as in section 4.1, the state $|\mathfrak{Z};\tau_\pm;\pm\rangle'$ evolves in the opposite time direction relative to his proper time $\tau_\pm$. Indeed, considering the time evolution operator $U_V(t) = V\exp[-\mathrm{i}\omega_0 t\mathbb{H}]V^\dagger$, we have the Gazeau–Klauder temporal stability condition for dual states, which is written as

$$U_V(t)|\mathfrak{z};\tau_\pm;\pm\rangle' = |\mathfrak{z};\tau_\pm - t;\pm\rangle'. \tag{172}$$

This shows that the dual state $|\mathfrak{z};\tau_\pm;\pm\rangle'$ is a kind of 'proper time reversal state' of its dual partner $|\mathfrak{z};\tau_\pm;\pm\rangle$. This can also be seen by considering the following proper Schrödinger equations:

$$(\mathrm{i}\hbar)\partial_{\tau_\pm}|\mathfrak{z};\tau_\pm;\pm\rangle = (\mathrm{i}\hbar)(-\mathrm{i}\tau_\pm\omega_0\mathbb{H})|\mathfrak{z};\tau_\pm;\pm\rangle, \tag{173}$$

$$(\mathrm{i}\hbar)\partial_{\tau_\pm}|\mathfrak{z};\tau_\pm;\pm\rangle' = (\mathrm{i}\hbar)(+\mathrm{i}\tau_\pm\omega_0\mathbb{H})|\mathfrak{z};\tau_\pm;\pm\rangle'. \tag{174}$$

Again, a similar treatment as above for finding solutions of the resolution of unity moment problems makes explicit dual matrix NVCSs. There are still a number of different displacement operators for NVCSs over matrix domains.

6.3. Deformed T operators

Ordinary concept of T operators. Types of operators allowing the mapping between canonical and NCSs have been highlighted by Ali *et al* [7] and definitely used in [6]. This mapping rests on the idea that one could transform CSs into deformed ones via an operator called T and into dual CSs via the inverse operator T^{-1}, with $TT^{-1} = \mathbb{I}$, the identity onto the Hilbert space, considered. We have, using ordinary CS notations,

$$|z\rangle \overset{T}{\longmapsto} |z\rangle_f, \qquad |z\rangle \overset{T^{-1}}{\longmapsto} |z\rangle'_f. \tag{175}$$

For the simple instance of canonical CSs and NCSs, one gets

$$\left[|z\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle\right] \overset{T}{\longmapsto} \left[|z\rangle_f = N(|z|)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{x_n!}}|n\rangle\right] \tag{176}$$

$$x_n! := \prod_{k=1}^{n} x_k, \qquad x_0! := 1, \qquad N(|z|) = \left[\sum_{n=0}^{\infty}\frac{|z|^{2n}}{x_n!}\right]^{-\frac{1}{2}}, \tag{177}$$

with x_n being a nonlinear function of n, i.e. the deformation function on which one should impose the condition such that the norm series $N(|z|)$ converges in a nonempty complex disk, and the operator

$$T = N(|z|)\,\mathrm{e}^{+\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\sqrt{\frac{n!}{x_n!}}|n\rangle\langle n|. \tag{178}$$

A rapid verification shows that T^{-1} is well defined and,

$$\tilde{T}^{-1} = N'(|z|)N(|z|)\,\mathrm{e}^{+\frac{1}{2}|z|^2}T^{-1}, \qquad N'(|z|) = \left[\sum_{n=0}^{\infty}\frac{|z|^{2n}}{(n!)^2}(x_n!)\right]^{-\frac{1}{2}}, \tag{179}$$

maps $|z\rangle$ onto the dual state $|z\rangle'_f$. Let us observe that, in foregoing studies, these T and T^{-1} do not include normalization factors [6]. This could bring ambiguities when one wants to map normalized CSs onto normalized deformed one. As an answer to this issue, the definitions of T (178) and $\tilde{T}^{-1}$ (179) provide the correct operators. Furthermore, the temporal stability axiom is not verified by any of the above-mentioned states. Let us find an improved formulation for canonical VCSs, NVCS and dual NVCSs allowing time translations.

Matrix T operators. We only deal with the case of normal matrix domains. One can easily deduce the results for quaternionic and for S^2-NVCSs by a similar analysis. Consider NVCSs over normal matrices, given $V \in U(2)$. Then, we define the operators

$$T_f = N(|\mathfrak{Z}|)(N_0(|\mathfrak{Z}|))^{-1} \sum_{n=0,\pm}^{\infty} |n\rangle\langle n| \otimes V\sqrt{n!}R_0(n)\,\mathrm{e}^{+\mathrm{i}\omega_0\tau(e_n^0-e_n)}V^\dagger|\pm\rangle\langle\pm| \tag{180}$$

$$N_0(|\mathfrak{Z}|) = \left[\sum_{n=0}^{\infty}\left(\frac{|z|^{2n}}{n!}+\frac{|w|^{2n}}{n!}\right)\right]^{-\frac{1}{2}}, \tag{181}$$

where $e_n^0 = \lim_{f(n)\to 1} e_n$ is the eigenenergies of the canonical model and $N_0(|\mathfrak{Z}|)$ is the normalization factor of the canonical Gazeau–Klauder VCSs

$$|\mathfrak{Z},\tau_\pm,\pm\rangle_0 = N_0(|\mathfrak{Z}|)\sum_{n=0}^{\infty}|n\rangle\otimes V\frac{1}{\sqrt{n!}}\,\mathrm{e}^{-\mathrm{i}\omega_0\tau e_n^0}\mathfrak{Z}_{\mathrm{diag}}^n V^\dagger|\pm\rangle, \tag{182}$$

which could be deduced from matrix NVCSs of section 5 by taking deformation parameters limit $(\kappa, f(N)) \to (1,1)$. The phase of T_f (180) will contribute to the temporal stability of the resulting state. A straightforward calculation gives the correct mapping of (time stable and normalized) canonical VCSs onto (time stable and normalized) NVCSs

$$T_f|\mathfrak{Z},\tau_\pm,\pm\rangle_0 = |\mathfrak{Z},\tau_\pm,\pm\rangle. \tag{183}$$

Next, let us seek the operator mapping $|\mathfrak{Z},\tau_\pm,\pm\rangle_0$ onto $|\mathfrak{Z},\tau_\pm,\pm\rangle'$. Regarding the inverse operator

$$T_f^{-1} = (N(|\mathfrak{Z}|))^{-1}N_0(|\mathfrak{Z}|)\sum_{n=0,\pm}^{\infty}|n\rangle\langle n|\otimes V\frac{1}{\sqrt{n!}}(R_0(n))^{-1}\,\mathrm{e}^{-\mathrm{i}\omega_0\tau(e_n^0-e_n)}|\pm\rangle\langle\pm|V^\dagger, \tag{184}$$

clearly this does not furnish the correct answer if we would respect all Gazeau–Klauder axioms. However, keeping in mind that dual states are proper time reversal states of the original theory, the operator $\mathcal{T}_f$ defined as

$$\mathcal{T}_f = (N(|\mathfrak{Z}|))^2(N_0(|\mathfrak{Z}|))^{-2}T_f^{-1} \tag{185}$$

gives the answer

$$\mathcal{T}_f|\mathfrak{Z},-\tau_\pm,\pm\rangle_0 = |\mathfrak{Z},\tau_\pm,\pm\rangle', \tag{186}$$

indicating that canonical proper time reversal VCSs are mapped onto dual NVCSs.

7. A new class of S^3 NVCSs

There is another class of exactly solvable NVCSs that could be defined on the Hilbert space $\mathcal{V}$ and still be continuous at $z = 0$. It is worth noting that although the previous construction including the finite sequence of states into one or another tower works well, another alternative could also be of interest. Indeed, the considered finite sequence of k-initial states of the Hilbert space can be viewed as a third part on its own, not depending on the two towers, to which one can assign a new vector index (here an angle parametrizing S^3). The resulting NVCSs satisfy Gazeau–Klauder properties and yield an exact solution of their moment problem associated with the resolution of the identity.

The following states, which we shall refer to S^3 NVCSs, are parametrized by the unit sphere S^3 vectors labeled by the angles (Θ, ϕ), $\Theta = (\theta_1, \theta_2)$, $\theta_i \in [0, \pi]$, $\phi \in [0, 2\pi[$, and the real time parameters $(\tau_*, \tau_\pm)$

$$\begin{aligned}|z; (\tau_*, \tau_\pm); (\Theta, \phi)\rangle = {}& \mathcal{N}^*(|z|) \cos\theta_1 \sum_{q=0}^{k-1} \frac{z^q}{K_-^0(\{q\})!} \,\mathrm{e}^{-\mathrm{i}\omega_0 \tau_* e_q^-} \left|e_q^-\right\rangle \\ &+ \mathcal{N}^-(|z|) \sin\theta_1 \cos\theta_2 \sum_{n=k}^{\infty} \frac{z^n}{K_-^0(\{n\})!} \,\mathrm{e}^{-\mathrm{i}\omega_0 \tau_- e_n^-} \left|e_n^-\right\rangle \\ &+ \mathcal{N}^+(|z|)\, \mathrm{e}^{\mathrm{i}\phi} \sin\theta_1 \sin\theta_2 \sum_{n=0}^{\infty} \frac{z^n}{K_+^0(\{n\})!} \,\mathrm{e}^{-\mathrm{i}\omega_0 \tau_+ e_n^+} \left|e_n^+\right\rangle, \end{aligned} \tag{187}$$

where the norm series,

$$\mathcal{N}^*(|z|) = \left\{ \sum_{q=0}^{k-1} \frac{|z|^{2q}}{\left(K_-^0(\{q\})!\right)^2} \right\}^{-\frac{1}{2}}, \tag{188}$$

$$\mathcal{N}^\pm(|z|) = \left\{ \sum_{n=n_0^\pm}^{\infty} \frac{|z|^{2n}}{\left(K_\pm^0(\{n\})!\right)^2} \right\}^{-\frac{1}{2}}, \qquad n_0^+ = 0, \quad n_0^- = k, \tag{189}$$

are such that $|z| \leqslant R$, $R = \min(R_+, R_-)$ being the minimum of the convergence radii $R_\pm = \lim_{n\to\infty} \left(K_\pm^0(\{n\}\right)$ of $\mathcal{N}^\pm(|z|)$. The positive functions $K_\pm^0(\{n\})$ are, for the moment, still free.

We sketch the proof that the S^3 NVCSs (187) are of the Gazeau–Klauder type:

(i) The normalization and continuity of labeling are clearly guaranteed after a simple evaluation.

(ii) The time evolution of these states under the unitary operator $U(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t \mathcal{H}^{\mathrm{red}}}$ is such that

$$U(t)|z; (\tau_*, \tau_\pm); (\Theta, \phi)\rangle = |z; (\tau_* + t, \tau_\pm + t); (\Theta, \phi)\rangle, \tag{190}$$

so the total set of S^3 NVCSs is stable under time evolution.

(iii) Action-angle variables are $(\{J_*, J_-, J_+\}, \{\tau_*, \tau_-, \tau_+\})$ such that

$$J_* = (\mathcal{N}^*(|z|))^2 (\cos\theta_1)^2 \sum_{q=0}^{k-1} \frac{|z|^{2q}}{\left(K_-^0(\{q\})!\right)^2} e_q^-, \tag{191}$$

$$J_- = (\mathcal{N}^-(|z|))^2 (\sin\theta_1 \cos\theta_2)^2 \sum_{n=k}^{\infty} \frac{|z|^{2n}}{\left(K_-^0(\{n\})!\right)^2} e_n^-, \tag{192}$$

$$J_+ = (\mathcal{N}^+(|z|))^2 (\sin\theta_1 \sin\theta_2)^2 \sum_{n=k}^{\infty} \frac{|z|^{2n}}{\left(K_+^0(\{n\})!\right)^2} e_n^+. \tag{193}$$

(iv) The resolution of the identity can be written as

$$\begin{aligned}\mathbb{I}_{\mathcal{V}} &= \sum_{n=0,\pm}^{\infty} \left|e_n^\pm\right\rangle\!\left\langle e_n^\pm\right| \qquad (194)\\ &= \int_{D_R \times S^3} \mathrm{d}\mu(z; \Theta, \phi) |z; (\tau_*, \tau_\pm); (\Theta, \phi)\rangle\langle z; (\tau_*, \tau_\pm); (\Theta, \phi)|, \qquad (195)\end{aligned}$$

where the S^3 measure $\mathrm{d}\mu(z;\Theta,\phi)$ has the parametrization

$$\mathrm{d}\mu(z;\theta,\phi) = \sin\theta_1 \sin\theta_2\, \mathrm{d}^2z\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2\, \mathrm{d}\phi \left\{ W^+(|z|) \sum_{n=0}^{\infty} |e_n^+\rangle\langle e_n^+| \right.$$
$$\left. + W^-(|z|) \sum_{n=k}^{\infty} |e_n^-\rangle\langle e_n^-| + W^*(|z|) \sum_{n=0}^{k-1} |e_n^-\rangle\langle e_n^-| \right\}. \tag{196}$$

Here $W^*(|z|)$ and $W^{\pm}(|z|)$ are real weight functions, which are yet unknown.

After substitution in (195), again with $z = r\exp(\mathrm{i}\varphi)$ and a measure in the radial sector chosen as $\mathrm{d}^2z = r\,\mathrm{d}r\,\mathrm{d}\varphi$ with $r \in [0, R)$ and $\varphi \in [0, 2\pi[$, one comes to the moment problems

$$0 \leqslant n \leqslant k-1, \qquad \int_0^{R^2} \mathrm{d}u\, u^n h_*(u) = \left(K_-^0(\{n\})!\right)^2, \tag{197}$$

$$n \geqslant n_0^{\pm}, \qquad \int_0^{R^2} \mathrm{d}u\, u^n h_{\pm}(u) = \left(K_{\pm}^0(\{n\})!\right)^2, \tag{198}$$

where $u = r^2$ and the moment functions $h_*(r^2)$ and $h_{\pm}(r^2)$ are such that

$$h_*(r^2) = \frac{8\pi^2}{3} |\mathcal{N}^*(r)|^2 W^*(r), \tag{199}$$

$$h_-(r^2) = \frac{16\pi^2}{9} |\mathcal{N}^-(r)|^2 W^-(r), \tag{200}$$

$$h_+(r^2) = \frac{32\pi^2}{9} |\mathcal{N}^+(r)|^2 W^+(r). \tag{201}$$

In order to solve the problems (197)–(201), we can set some constraints onto the free deformation function $K_{\pm}^0(\{n\})$. Let us observe some solutions in the undeformed situation. We then map $(\kappa, f(N)) \to (1, 1)$, and set the so-called action-identity constraint [28] defined by the set of relations

$$J_* = \cos^2\theta_1 \left(|z|^2 + e_0^-\right), \tag{202}$$

$$J_- = \sin^2\theta_1 \cos^2\theta_2 \left(|z|^2 + e_k^-\right), \tag{203}$$

$$J_+ = \sin^2\theta_1 \sin^2\theta_2 \left(|z|^2 + e_0^+\right). \tag{204}$$

We can infer from (202)–(204), at the decoupled model limit $\lambda(N) \to 0$, the constraints

$$K_-^0(n) = \sqrt{e_n^- - e_0^-} = \sqrt{(1+\epsilon)n}, \qquad 0 \leqslant n \leqslant k-1, \tag{205}$$

$$K_-^0(n) = \sqrt{e_n^- - e_k^-} = \sqrt{(1+\epsilon)n}, \qquad n \geqslant k, \tag{206}$$

$$K_+^0(n) = \sqrt{e_n^+ - e_0^+} = \sqrt{(1+\epsilon)n}, \qquad n \geqslant 0, \tag{207}$$

assuming a bounded from below energy spectrum, i.e. $e_n^{\pm} - e_0^{\pm} \geqslant 0$. The subsequent norm series,

$$|\mathcal{N}^*(r)|^{-2} = \sum_{q=0}^{k-1} \frac{|z|^{2q}}{(1+\epsilon)^q q!}, \tag{208}$$

$$|\mathcal{N}^{-}(r)|^{-2} = \mathrm{e}^{\frac{r^2}{1+\epsilon}} - |\mathcal{N}^{*}(r)|^{-2}, \tag{209}$$

$$|\mathcal{N}^{+}(r)|^{-2} = \mathrm{e}^{\frac{r^2}{1+\epsilon}}, \tag{210}$$

are of infinite radii of convergence. The moment problems can easily be performed with solutions $h^*(r) = h^{\pm}(r) = \exp[-r^2/(1+\epsilon)]/(1+\epsilon)$, from which we can deduce the weights

$$W^*(r) = \frac{3\mathrm{e}^{-\frac{r^2}{1+\epsilon}}}{8\pi^2(1+\epsilon)}|\mathcal{N}^*(r)|^{-2} \tag{211}$$

$$W^-(r) = \frac{9}{16\pi^2(1+\epsilon)}\left(1 - \mathrm{e}^{-\frac{r^2}{1+\epsilon}}|\mathcal{N}^*(r)|^{-2}\right), \tag{212}$$

$$W^+(r) = \frac{9}{32\pi^2(1+\epsilon)}, \tag{213}$$

indicating a new class of NVCSs.

We can turn the discussion to the deformed case. If we set $K^0_{\pm}(\{n\}) = \sqrt{\{n\}}$, we end up with the moment problems

$$0 \leqslant n \leqslant k-1, \qquad \int_0^{R^2} \mathrm{d}u\, u^n h_*(u) = \{n\}!, \tag{214}$$

$$n \geqslant n^0_{\pm}, \qquad \int_0^{R^2} \mathrm{d}u\, u^n h_{\pm}(u) = \{n\}!, \tag{215}$$

with $n^0_- = k, n^0_+ = 0$, whose solutions can be provided in terms of (p,q) deformations as previously performed.

8. Conclusion

The construction of new Gazeau–Klauder-type NVCSs for spin–orbit Hamiltonians has been achieved in this work. We have extended the action of the ladder operators to the initial finite-dimensional set of states related to the multiphoton processes. We have also succeeded in finding exact solutions to the resolution of the identity for different sets of NVCSs. Besides, we have addressed the issues of different displacement and T operators which generate the variety of states that we have found. Moreover, we have built a new class of NVCSs parametrized by unit vectors of the S^3 sphere and proved that the latter also generate an overcomplete set of VCSs. Finally, it is worth emphasizing that Gazeau–Klauder axioms are nonempty in the full deformation theory.

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Appendix

This appendix lists useful identities on (p, q)-deformed exponential functions. We use the notations and convention of [11] with (p, q)-shifted products and factorials defined as, for any real parameters a, b and α such that $a \neq 0$, $p > 1$, $0 < q < 1$ and $pq < 1$,

$$[a, b; p, q]_0 = 1, \qquad [a, b; p, q]_\alpha = \frac{[a, b; p, q]_\infty}{[ap^\alpha, bq^\alpha; p, q]_\infty},$$
$$[a, b; p, q]_\infty = \prod_{n=0}^{\infty} \left(\frac{1}{ap^n} - bq^n \right). \tag{A.1}$$

Given new parameters $(z, \mu, \nu) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}$, the usual exponential function e^z, $z \in \mathbb{C}$, can be extended to the generalized (μ, ν, p, q)-exponential as follows

$$\mathcal{E}^{(\mu,\nu)}_{(p,q)}(z) = \sum_{n=0}^{\infty} \left(\frac{q^\mu}{p^\nu} \right)^{n^2} \frac{z^n}{[p, q; p, q]_n}, \tag{A.2}$$

provided $q^{2\mu} p^{1-2\nu} \leqslant 1$. The exponential function is recovered after rescaling $z \to z(p^{-1} - q)$, for example, and taking the limit $\lim_{(p,q)\to(1,1)} \mathcal{E}^{\mu,\nu}_{(p,q)}(z(p^{-1} - q)) = \mathrm{e}^z$. Through the reduction $\mu = 0$ and $\nu = 1/2$, (A.2) generates another (p, q)-exponential as

$$e_{(p,q)}(z) = \sum_{n=0}^{\infty} \frac{1}{p^{n^2/2}} \frac{z^n}{[p, q; p, q]_n}, \qquad |z| < p^{-1/2}. \tag{A.3}$$

The next identity stands for the (p, q)-analog of the Euler Gamma function, i.e. the (p, q)-analog of the Ramanujan q-integral, for any $n \in \mathbb{N}$,

$$\int_0^\infty \mathrm{d}t\, t^n e_{(p,q)}(-\lambda_0 p^{-1/2} t) = \frac{[p, q; p, q]_n}{\lambda_0^{n+1} q^{n(n+1)/2}} \log\left(\frac{1}{pq} \right). \tag{A.4}$$

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